

# Groundwork for Hodge-Refined Whitehead Torsion

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## Abstract

Given a manifold  $M$  of dimension  $n \geq 5$ , the  $s$ -cobordism theorem provides a bijective correspondence between  $h$ -cobordisms of  $M$  and elements of  $\mathrm{Wh}(\pi_1(M)) := K_1(\mathbb{Z}[\pi_1(M)])/\{\pm\pi_1(M)\}$  through a cobordism invariant known as Whitehead torsion. Let  $X$  be a smooth projective variety over  $\mathbb{C}$ , and let  $W$  be an algebraic deformation of  $X$  parameterized by the unit interval  $I$  such that the fibers of  $W$  have isomorphic (mixed) Hodge structures on their cohomology and quotients of  $\mathbb{Q}[\pi_1(X)]$  by powers of its augmentation ideal. In this paper we lay some of the groundwork for defining a variation of Whitehead torsion which can detect whether an algebraic deformation of such a variety is trivial. Since we expect this invariant to make use of various Hodge structures present for smooth projective varieties, we refer to this invariant as “Hodge-refined Whitehead torsion.” We first provide an explicit computation of the Hodge structures on the cohomology of a class of smooth projective varieties with fundamental group  $\mathbb{Z}/5\mathbb{Z}$ . In doing so, we develop a spectral sequence for calculating the Hodge structures and numbers of projective hypersurfaces and their quotients by free group actions, and we explore some of its consequences. Finally, we define a notion of “Hodge-refined” rings and modules, and we discuss how we expect these tools to be used in defining a Hodge-refined Whitehead torsion.

## Motivation and Setup

The Whitehead group of a group  $G$ , denoted  $\mathrm{Wh}(G)$ , is defined to be  $K_1(\mathbb{Z}[G])/\{\pm G\}$ , where  $K_1(\mathbb{Z}[G])$  denotes the first algebraic  $K$ -group of  $\mathbb{Z}[G]$ . Unless otherwise stated, all varieties are assumed to be over  $\mathbb{C}$ . We will often use without mention the fact that smooth projective varieties admit a smooth and complex manifold structure.

Given a closed, connected smooth manifold of dimension  $\geq 5$ , the  $s$ -cobordism theorem asserts that there is a bijective correspondence between diffeomorphism class short exact sequence of  $h$ -cobordisms  $(M, N, W, i, j)$  and elements of the Whitehead group  $\mathrm{Wh}(\pi_1(M))$  ([1, Theorem 2.1]). This correspondence is given by a diffeomorphism invariant known as the Whitehead torsion of an  $h$ -cobordism.

Let  $\mathcal{M}$  be a smooth manifold such that each point is a connected smooth projective variety over  $\mathbb{C}$  of degree and dimension  $n$ , and let  $\gamma : I \rightarrow \mathcal{M}$  be a smooth path. Then we may equip  $W := \bigsqcup_{t \in I} \gamma(t)$  with a topology and smooth structure so that  $W$  is a compact connected manifold with boundary of dimension  $n + 1$ . We call  $W$  an *algebraic deformation of  $\gamma(0)$  parameterized by  $I$* . Note that there is an obvious proper surjective submersion  $W \rightarrow I$ , so it follows from Ehresmann’s lemma that this map is a fiber bundle, and since  $I$  is contractible,  $W \cong \gamma(0) \times I$  as smooth fiber bundles. A morphism of deformations  $W_1 \rightarrow W_2$  is a bundle map from  $W_1 \rightarrow I$  to  $W_2 \rightarrow I$  such that the restrictions to each fiber are maps of smooth projective varieties. A *trivial algebraic deformation* is a deformation isomorphic (as deformations) to one with  $\gamma$  constant. Since two smooth projective varieties may be isomorphic as smooth manifolds but not as varieties, deformation isomorphisms are stronger

than diffeomorphisms. From this we see that all deformations are trivial  $h$ -cobordisms, but they also contain algebraic information, and thus may be viewed as a sort of “algebraic-geometric” refinement of cobordisms.

With this in mind and with the  $s$ -cobordism theorem as inspiration, we wish to define an analog of the Whitehead torsion for deformations of smooth projective varieties. In particular, given a smooth projective variety  $X$ , we wish to define an abelian group  $\Gamma$  depending on  $\pi_1(X)$ , and to each algebraic deformation  $W$  an element  $g \in \Gamma$  such that  $g = 0$  iff  $W$  is a trivial deformation. It is a well-known fact that the  $n$ -th cohomology of a smooth projective variety carries a mixed Hodge structure of weight  $n$ . Furthermore, as demonstrated in [2], for each  $s \geq 1$  the truncation  $\mathbb{Q}[\pi_1(X)]/\mathcal{J}^s$  admits a mixed Hodge structure. Here  $\mathcal{J}$  is the augmentation ideal of  $\mathbb{Q}[\pi_1(X)]$ . We are working to define the aforementioned invariant by making use of these Hodge structures—for this reason we refer to the invariant as *Hodge-refined Whitehead torsion*. In order to try and keep the construction of this invariant as close to that of Whitehead torsion as possible, we will assume that given a deformation  $W$ , the Hodge structures on the cohomology and truncated group ring of each fiber of  $W \rightarrow I$  are isomorphic.

Consider the following setup. Let  $G = \mathbb{Z}/5\mathbb{Z} = \langle \sigma \rangle$  act on  $\mathbb{C}[x_0, x_1, x_2, x_3]$  by  $\sigma.x_i := \zeta^i x_i$ , where  $\zeta$  is a primitive fifth root of unity. This gives rise to an action of  $\mathbb{Z}/5\mathbb{Z}$  on  $\mathbb{P}^3$ . Consider now the space of  $G$ -symmetric degree 5 polynomials. Then  $\mathbb{Z}/5\mathbb{Z}$  acts on the projectivized zero locus of each of these polynomials. Let  $\mathcal{H}$  be the space of these projectivized zero loci on which  $\mathbb{Z}/5\mathbb{Z}$  acts freely. We will use  $X$  to denote such a hypersurface; An example of such a hypersurface is that carved out by  $f := x_0^5 + x_1^5 + x_2^5 + x_3^5$ . Since the group action is free,  $X/G$  will be a smooth projective variety. Furthermore, the hypersurfaces are simply-connected, thus  $X/G$  will have fundamental group  $\mathbb{Z}/5\mathbb{Z}$ . Taking a deformation  $W$  through  $\mathcal{H}$  of a hypersurface  $X$  in  $\mathcal{H}$ , note that  $G$  acts on each fiber freely, and thus acts on  $W$  freely. Furthermore, the map  $W \rightarrow I$  factors through  $W/G \rightarrow I$ , and each fiber of the latter map is a smooth projective variety. It follows that  $W/G$  is a deformation of  $X/G$ . It is these types of deformations which we will be using as the “toy example” with which we plan to build a Hodge-refined Whitehead torsion.

In this paper, we provide preliminary results and definitions which we expect to use in defining Hodge-refined Whitehead torsion. In Section 1, we compute the Hodge structures on the cohomology of the hypersurfaces  $X$  described above. Section 2 develops several tools for computing Hodge structures and numbers of quotients of smooth projective varieties by group actions. Section 3 applies the results of Sections 1 and 2 to the example listed above. In Section 4, we define a notion of “Hodge-refined” rings and modules. We hope to define  $\Gamma$  in a way similar to the first  $K$ -group of a ring using these “Hodge-refined” objects. Finally, we discuss how we expect to use these tools in defining a Hodge-refined Whitehead torsion.

## 1. Hodge Numbers of Degree $d$ Hypersurface in $\mathbb{P}^3$

Throughout this section, let  $k$  denote an arbitrary field and  $X$  a smooth projective variety. The Hodge numbers  $h_X^{p,q}$  of a smooth projective variety  $X$  are the  $k$ -dimension of the sheaf cohomology  $H^q(X, \Omega_X^p)$ . Here  $\Omega_X^p$  denotes the sheaf of Kähler  $p$ -differentials on  $X$ .

### Lemma 1.1

Let  $i : X \rightarrow Y$  be an inclusion of ringed spaces. Then  $i^*i_*\mathcal{O}_X \cong \mathcal{O}_X$ .

*Proof:*

There is an adjunction  $i^* \dashv i_*$  and thus a diagram of the form

$$\begin{array}{ccc} i^* \mathcal{O}_Y & \longrightarrow & i^* i_* i^* \mathcal{O}_Y \\ & \searrow & \downarrow \\ & & i^* \mathcal{O}_Y \end{array}$$

Since  $i^* \mathcal{O}_Y \cong \mathcal{O}_X$ , we have a surjection  $i^* i_* \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X$ . Since  $i$  is an inclusion, for  $x \in X$  we have  $(i_* \mathcal{O}_X)_x \cong \mathcal{O}_{X,x}$ . Taking stalks of the map  $i^* i_* \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X$ , we have for all  $x \in X$  a surjection

$$(i^* i_* \mathcal{O}_X)_x = (i_* \mathcal{O}_X)_x = \mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}$$

But  $\mathcal{O}_{X,x}$  is projective in  $\mathcal{O}_{X,x}\text{-mod}$ , so letting  $I$  denote the kernel of the map above,  $\mathcal{O}_{X,x} \cong \mathcal{O}_{X,x} \oplus I$ . But since  $\mathcal{O}_{X,x}$  is commutative,  $I = 0$ , thus the map on stalks is an isomorphism, thus  $i^* i_* \mathcal{O}_X \rightarrow \mathcal{O}_X$  is an isomorphism. ■

### Proposition 1.1

Let  $X$  be a nonsingular hypersurface of  $\mathbb{P}_k^n$  of degree  $d$  with  $n > 2$ . Then for all  $p + q \neq n - 1$

$$H^q(X, \Omega_X^p) \cong \begin{cases} k & \text{if } p = q \text{ and } p \leq n - 1 \\ 0 & \text{otherwise} \end{cases}$$

*Proof:*

We first show that if  $X \subseteq \mathbb{P}_k^n$  is nonsingular of degree  $d$ , then for  $p + q < n - 1$  we have for each  $m \in \mathbb{Z}_{\geq 0}$

$$H^q(X, \Omega_X^p(-m)) \cong H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-m))$$

We proceed by induction on  $p$ , starting with the case that  $p = 0$ , where we interpret  $\Omega^0$  to be the structure sheaf  $\mathcal{O}$ . Since the hypersurface  $X = V_+(f)$  for some homogeneous polynomial  $f \in k[x_0, \dots, x_n]$  of degree  $d$ , there is a short exact sequence

$$0 \rightarrow k[x_0, \dots, x_n](-d) \rightarrow k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]/(f) \rightarrow 0$$

Which induces for each  $m \in \mathbb{Z}_{\geq 0}$  a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d - m) \rightarrow \mathcal{O}_{\mathbb{P}}(-m) \rightarrow i_* \mathcal{O}_X(-m) \rightarrow 0$$

From which we obtain the following exact sequences for all  $p \in \mathbb{Z}_{\geq 0}$

$$H^q(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}}(-d - m)) \rightarrow H^q(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}}(-m)) \rightarrow H^q(X, \mathcal{O}_X(-m)) \rightarrow H^{q+1}(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}}(-d - m))$$

By [3],  $H^\ell(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}}(-d - m)) = 0$  whenever  $\ell \neq n$ , thus for  $q + 1 < n$ , the exact sequence above implies

$$H^p(X, \mathcal{O}_X(-m)) \cong H^p(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}}(-m))$$

Which completes the base case.

We now do some preparatory work before completing the induction. The inclusion map  $i : X \rightarrow \mathbb{P}_k^n$  is a closed immersion, thus there is a surjection  $\mathcal{O}_{\mathbb{P}} \rightarrow i_* \mathcal{O}_X$ . Let  $\mathcal{J}$  denote the kernel

of this map. Since  $\mathcal{O}_X\text{-mod}$  is abelian,  $i_*\mathcal{O}_X \cong \mathcal{O}_{\mathbb{P}}/\mathcal{J}$ . Because  $X$  is nonsingular, the inclusion map  $i$  is smooth, so by [4, Theorem 8.17], the following is exact

$$0 \rightarrow i^*(\mathcal{J}/\mathcal{J}^2) \rightarrow i^*\Omega_{\mathbb{P}} \rightarrow \Omega_X \rightarrow 0$$

Then

$$i^*(\mathcal{J}/\mathcal{J}^2) \cong i^*(\mathcal{J} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}/\mathcal{J}) \cong i^*\mathcal{J} \otimes_{i^*\mathcal{O}_{\mathbb{P}}} i^*(\mathcal{O}_{\mathbb{P}}/\mathcal{J}) \cong i^*\mathcal{J} \otimes_{\mathcal{O}_X} i^*i_*\mathcal{O}_X \cong i^*\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong i^*\mathcal{J}$$

Where the penultimate isomorphism follows from Lemma 1.1. Note that the short exact sequence  $0 \rightarrow \mathcal{O}_{\mathbb{P}}(-d) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow i_*\mathcal{O}_X \rightarrow 0$  implies  $\mathcal{J} \cong \mathcal{O}_{\mathbb{P}}(-d)$ . Because  $i^*\mathcal{O}_{\mathbb{P}}(-d) \cong \mathcal{O}_X(-d)$ , the short exact sequence above becomes

$$0 \rightarrow \mathcal{O}_X(-d) \rightarrow i^*\Omega_{\mathbb{P}} \rightarrow \Omega_X \rightarrow 0$$

Since  $\mathcal{O}_X(-d)$  is locally free of rank 1, by [5, Corollary 0.3] there is a short exact sequence for  $q \in \mathbb{Z}_{\geq 1}$  given by

$$0 \rightarrow \Omega_X^{p-1}(-d) \rightarrow i^*\Omega_{\mathbb{P}}^p \rightarrow \Omega_X^p \rightarrow 0$$

And thus for any  $m \in \mathbb{Z}_{\geq 0}$  and  $p \in \mathbb{Z}_{\geq 1}$  the following is exact

$$0 \rightarrow \Omega_X^{p-1}(-d-m) \rightarrow i^*\Omega_{\mathbb{P}}^p(-m) \rightarrow \Omega_X^p(-m) \rightarrow 0$$

Now suppose for sake of induction that whenever  $p \leq \ell - 1$ , we have for  $p + q < n - 1$  and any  $m \in \mathbb{Z}_{\geq 0}$  that

$$H^q(X, \Omega_X^p(-m)) \cong H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-m))$$

Now let  $p = \ell$ . Using the fact that  $X$  is closed in  $\mathbb{P}_k^n$  and thus  $H^q(X, i^*\Omega_{\mathbb{P}}^p(-m)) \cong H^q(\mathbb{P}_k^n, i_*i^*\Omega_{\mathbb{P}}^p(-m))$ , the short exact sequence from above yields for each  $q \in \mathbb{Z}_{\geq 0}$  the exact sequence

$$\begin{aligned} H^q(X, \Omega_X^{p-1}(-d-m)) &\rightarrow H^q(\mathbb{P}_k^n, i_*i^*\Omega_{\mathbb{P}}^p(-m)) \\ &\rightarrow H^q(X, \Omega_X^p(-m)) \rightarrow H^{q+1}(X, \Omega_X^{p-1}(-d-m)) \end{aligned}$$

By the induction hypothesis, when  $(q+1) + (p-1) = p+q < n-1$ , the outer maps are isomorphic to  $H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^{p-1}(-d-m))$  and  $H^{q+1}(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^{p-1}(-d-m))$ . Since  $0 \leq p, q < n-1$ , so by [6, Theorem 17.1.5 (Bott)], these are zero, thus for  $p+q < n-1$  and any  $m \in \mathbb{Z}_{\geq 0}$

$$H^q(X, \Omega_X^p(-m)) \cong H^q(\mathbb{P}_k^n, i_*i^*\Omega_{\mathbb{P}}^p(-m))$$

By [6, Lemma 1.17.2.3], there is a SES

$$0 \rightarrow \Omega_{\mathbb{P}}^p(-d-m) \rightarrow \Omega_{\mathbb{P}}^p(-m) \rightarrow i_*i^*\Omega_{\mathbb{P}}^p(-m) \rightarrow 0$$

Which gives for each  $q \in \mathbb{Z}_{\geq 0}$  the exact sequence

$$H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-d-m)) \rightarrow H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-m)) \rightarrow H^q(\mathbb{P}_k^n, i_*i^*\Omega_{\mathbb{P}}^p(-m)) \rightarrow H^{q+1}(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-d-m))$$

By [6, Theorem 17.1.5 (Bott)],  $H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-d-m)) = 0$  whenever  $q \neq n$ . Thus for  $q+1 < n$ , we find

$$H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-m)) \cong H^q(\mathbb{P}_k^n, i_*i^*\Omega_{\mathbb{P}}^p(-m))$$

Putting together our observations, we see that for  $p+q < n-1$  and any  $m \in \mathbb{Z}_{\geq 0}$

$$H^q(X, \Omega_X^p(-m)) \cong H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p(-m))$$

Which completes the induction step.

It is a well-known fact that

$$H^q(\mathbb{P}_k^n, \Omega_{\mathbb{P}}^p) \cong \begin{cases} k & \text{if } p = q \text{ and } q < n \\ 0 & \text{otherwise} \end{cases}$$

From which it immediately follows that for  $p + q < n - 1$

$$H^q(X, \Omega_X^p) \cong \begin{cases} k & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

By Serre duality ([4, Chapter III, Corollary 7.13]), since  $\dim X = n - 1$ , we have for each  $p, q \in \{0, \dots, n\}$

$$H^q(X, \Omega_X^p) \cong H^{n-1-q}(X, \Omega_X^{n-1-p})^\vee$$

Thus whenever  $(n - q - 1) + (n - p - 1) < n - 1$  and  $p, q \in \{0, \dots, n - 1\}$ ,

$$H^q(X, \Omega_X^p) \cong \begin{cases} k & \text{if } n - 1 - p = n - 1 - q \\ 0 & \text{otherwise} \end{cases}$$

But this is exactly the statement that for  $p + q > n - 1$  with  $p, q \in \{0, \dots, n - 1\}$

$$H^q(X, \Omega_X^p) \cong \begin{cases} k & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

Finally, by [4, Theorem 2.7], since  $X = V_+(f) \cong \text{Proj}(k[x_0, \dots, x_n]/(f))$  is Noetherian, for  $q > n - 1$  we have that for any  $p \in \mathbb{Z}_{\geq 0}$

$$H^q(X, \Omega_X^p) = 0$$

Putting all observations together, it follows that for  $p + q \neq n - 1$

$$H^q(X, \Omega_X^p) \cong \begin{cases} k & \text{if } p = q \text{ and } q \leq n - 1 \\ 0 & \text{otherwise} \end{cases}$$

■

## Proposition 1.2

The Hodge diamond for a degree 5 nonsingular hypersurface  $X$  in  $\mathbb{P}_k^3$  with  $k$  a characteristic zero field is given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 4 & & 45 & & 4 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

*Proof:*

From Proposition 1.1, we immediately have the following Hodge diamond for  $X$

$$\begin{array}{ccccc}
& & 1 & & \\
& & 0 & & 0 \\
h^{20} & & h^{11} & & h^{02} \\
& & 0 & & 0 \\
& & 1 & & 
\end{array}$$

Since  $X$  is degree 5, we have the SES

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-5) \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow i_* \mathcal{O}_X \rightarrow 0$$

Thus we have the exact sequence

$$H^2(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) \rightarrow H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}})$$

From [3],  $H^i(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}) = 0$  for  $i \neq 0$ , and

$$H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) = \left( \frac{1}{wxyz} k \left[ \frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right] \right)_{-5} \cong k^4$$

Putting these together, we see  $H^2(X, \mathcal{O}_X) \cong k^4$ , so  $h^{02} = 4$ . Since  $k$  is characteristic zero, we have from Hodge symmetry that  $h^{20} = 4$ .

It remains to compute the dimension of  $H^1(X, \Omega_X^1)$ . Using the short exact sequence  $0 \rightarrow \mathcal{O}_X(-5) \rightarrow i^* \Omega_{\mathbb{P}} \rightarrow \Omega_X \rightarrow 0$ , we have the exact sequence

$$\begin{aligned}
0 &= H^1(X, \mathcal{O}_X(-5)) \rightarrow H^1(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}}) \\
&\rightarrow H^1(X, \Omega_X) \rightarrow H^2(X, \mathcal{O}_X(-5)) \rightarrow H^2(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}}) \rightarrow H^2(X, \Omega_X) = 0
\end{aligned}$$

From  $0 \rightarrow \Omega_{\mathbb{P}}(-5) \rightarrow \Omega_{\mathbb{P}} \rightarrow i_* i^* \Omega_{\mathbb{P}} \rightarrow 0$  and the fact that  $H^1(\mathbb{P}_k^3, \Omega_{\mathbb{P}}(-5)) = H^2(\mathbb{P}_k^3, \Omega_{\mathbb{P}}(-5)) = H^2(\mathbb{P}_k^3, \Omega_{\mathbb{P}}) = H^3(\mathbb{P}_k^3, \Omega_{\mathbb{P}}) = 0$  by [6, Theorem 17.1.5 (Bott)], we find

$$H^1(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}}) \cong H^1(\mathbb{P}_k^3, \Omega_{\mathbb{P}}) \cong k$$

$$H^2(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}}) \cong H^3(\mathbb{P}_k^3, \Omega_{\mathbb{P}}(-5))$$

The short exact sequence  $0 \rightarrow \mathcal{O}_{\mathbb{P}}(-10) \rightarrow \mathcal{O}_{\mathbb{P}}(-5) \rightarrow i_* \mathcal{O}_X(-5) \rightarrow 0$  yields

$$\begin{aligned}
0 &= H^2(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) \rightarrow H^2(X, \mathcal{O}_X(-5)) \\
&\rightarrow H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-10)) \rightarrow H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) \rightarrow H^3(X, \mathcal{O}_X(-5)) = 0
\end{aligned}$$

Finally, the short exact sequence  $0 \rightarrow \Omega_{\mathbb{P}}(-5) \rightarrow \mathcal{O}_{\mathbb{P}}(-1)^4(-5) \rightarrow \mathcal{O}_{\mathbb{P}}(-5) \rightarrow 0$  ([4, Theorem 8.13]) and the fact that  $\mathcal{O}_{\mathbb{P}}(-1)^4(-5) \cong \mathcal{O}_{\mathbb{P}}(-1)(-5)^4 \cong \mathcal{O}_{\mathbb{P}}(-6)^4$  give the exact sequence

$$\begin{aligned}
0 &= H^2(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) \rightarrow H^3(\mathbb{P}_k^3, \Omega_{\mathbb{P}}(-5)) \rightarrow H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-6))^4 \\
&\rightarrow H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) \rightarrow H^4(\mathbb{P}_k^3, \Omega_{\mathbb{P}}(-5)) = 0
\end{aligned}$$

Using the fact that the  $k$ -dimension respects exact sequences, we find from the observations above that

$$h^{11} = 1 + \dim_k H^2(X, \mathcal{O}_X(-5)) - \dim_k H^2(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}})$$

$$\dim_k H^2(X, \mathcal{O}_X(-5)) = \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-10)) - \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5))$$

$$\begin{aligned}\dim_k H^2(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}}) &= \dim_k H^3(\mathbb{P}_k^3, \Omega_{\mathbb{P}}(-5)) \\ &= 4 \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-6)) - \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5))\end{aligned}$$

Putting everything together, we find

$$h^{11} = 1 + \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-10)) - 4 \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-6))$$

By [3], for  $d > 0$  we have

$$H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-d)) = \left( \frac{1}{wxyz} k \left[ \frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right] \right)_{-d}$$

Computing the dimension of this for  $d = 6$  and  $d = 10$  (see Appendix), we find

$$\dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-10)) = 84$$

$$\dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-6)) = 10$$

Therefore

$$h^{11} = 1 + 84 - 4(10) = 45$$

Which completes the Hodge diamond for  $X$ . ■

In the next section, we derive a spectral sequence for computing Hodge structures on quotients of smooth projective varieties by free group actions. While we will not make use of every result in this paper, we include them for sake of completeness.

## 2. Spectral Sequence for Quotients by Group Actions

Let  $X$  be a scheme over a characteristic zero field  $k$ , let  $G$  be a finite group acting on  $X$ , and let  $\mathcal{F}$  be a  $G$ -equivariant quasi-coherent  $\mathcal{O}_X$ -module. This induces a group action on the cohomologies  $H^i(X, \mathcal{F})$ . Take  $g \in G$ . Since  $\mathcal{F}$  is  $G$ -equivariant, there is an isomorphism  $\mathcal{F} \rightarrow g_* \mathcal{F}$ . Applying  $\Gamma_X$ , this induces an isomorphism  $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, g_* \mathcal{F}) = \Gamma(X, \mathcal{F})$ . The equivariance of  $\mathcal{F}$  implies that is a well-defined action, thereby making  $\Gamma(X, \mathcal{F})$  into a  $kG$ -module.

Consider the category  $\mathbf{QCoh}^G(\mathcal{O}_X)$  of  $G$ -equivariant quasi-coherent  $\mathcal{O}_X$ -modules; it is well-known that this is abelian and has enough injectives. Take an injective resolution  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^\bullet$  in this category. By abstract nonsense, the isomorphisms  $\mathcal{F} \rightarrow g_* \mathcal{F}$  extend uniquely to isomorphisms  $[0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^\bullet] \rightarrow [0 \rightarrow g_* \mathcal{F} \rightarrow g_* \mathcal{I}^\bullet]$  in the homotopy category  $\mathbb{K}^+(\mathbf{QCoh}^G(\mathcal{O}_X))$  of bounded below chain complexes. Note that  $\Gamma(X, g_* \mathcal{I}^\bullet) = \Gamma(X, \mathcal{I}^\bullet)$  and therefore  $0 \rightarrow g_* \mathcal{F} \rightarrow g_* \mathcal{I}^\bullet$  is a  $\Gamma_X$ -acyclic resolution of  $g_* \mathcal{F}$ , so applying  $\Gamma_X$  to  $[0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^\bullet]$  and  $[0 \rightarrow g_* \mathcal{F} \rightarrow g_* \mathcal{I}^\bullet]$  and taking cohomology yields isomorphisms  $R^i \Gamma_X(\mathcal{F}) \rightarrow R^i \Gamma_X(g_* \mathcal{F})$  associated to each  $g \in G$ . Since the construction is functorial, the  $G$ -equivariance of  $\mathcal{F}$  implies these isomorphisms define a  $G$ -action on  $R^i \Gamma_X(\mathcal{F}) = H^i(X, \mathcal{F})$  for each  $i \geq 0$ . Since  $X$  is a scheme over  $k$  and  $\mathcal{F}$  a  $\mathcal{O}_X$ -module, the  $H^i(X, \mathcal{F})$  can be realized as  $k$ -vector spaces and thus as  $kG$ -modules. Consequently, we can apply the group invariants functor  $(-)^G$  to get the  $k$ -vector spaces  $H^i(X, \mathcal{F})^G$ .

The  $G$ -equivariant structure on  $\mathcal{F}$  yields for each  $g \in G$  an isomorphism  $\mathcal{F} \rightarrow g_* \mathcal{F}$ . Assume  $G$  acts freely on  $X$ , and let  $\pi : X \rightarrow X/G$  denote the quotient map. Then  $\pi \circ g = \pi$ , and we

obtain an isomorphism  $\pi_*\mathcal{F} \rightarrow \pi_*g_*\mathcal{F} = (\pi \circ g)_*\mathcal{F} = \pi_*\mathcal{F}$ . The cocycle condition on the equivariant structure implies the maps  $\pi_*\mathcal{F} \rightarrow \pi_*\mathcal{F}$  associated to each  $g \in G$  are compatible, thus we obtain group actions on the sections of  $\pi_*\mathcal{F}$ . Define  $(\pi_*\mathcal{F})^G$  by  $\Gamma(U, (\pi_*\mathcal{F})^G) = \Gamma(U, \pi_*\mathcal{F})^G$ . Note that the action on  $\Gamma(X/G, \pi_*\mathcal{F})$  agrees with the action on  $\Gamma(X, \mathcal{F})$  defined in the previous paragraph, thus  $\Gamma(X/G, \pi_*\mathcal{F})^G = \Gamma(X, \mathcal{F})^G$ .

Let  $\mathbf{QCoh}(\mathcal{O}_{X/G})[G]$  denote the category of quasi-coherent  $\mathcal{O}_{X/G}$ -modules equipped with a group action. That is, objects are  $\mathcal{O}_{X/G}$ -modules  $\mathcal{F}$  with a collection of isomorphisms for each element of  $G$  such that the identity element of  $G$  corresponds to the identity morphism, and the following commutes for all  $g, h \in G$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{h} & \mathcal{F} \\ gh \downarrow & \swarrow g & \\ \mathcal{F} & & \end{array}$$

And morphisms are  $\mathcal{O}_X$ -module maps  $\tau : \mathcal{F} \rightarrow \mathcal{G}$  such that the following commutes for all  $g \in G$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{g} & \mathcal{F} \\ \tau \downarrow & & \downarrow \tau \\ \mathcal{G} & \xrightarrow{g} & \mathcal{G} \end{array}$$

Note that this induces a group action on each  $\Gamma(U, \mathcal{F})$  for  $U \in \mathcal{O}_{X/G}$ , so we can define a functor  $(-)^G : \mathbf{QCoh}(\mathcal{O}_{X/G})[G] \rightarrow \mathbf{QCoh}(\mathcal{O}_{X/G})$  by  $\Gamma(U, \mathcal{F}^G) = \Gamma(U, \mathcal{F})^G$ . Using the fact that  $\mathbf{QCoh}(\mathcal{O}_{X/G})$  is an abelian category, it can be checked that  $\mathbf{QCoh}(\mathcal{O}_{X/G})[G]$  is an abelian subcategory of  $\mathbf{QCoh}(\mathcal{O}_{X/G})$ .

### Theorem 2.1

*With the notation as above, there is a first quadrant spectral sequence*

$$H^i(X/G, R^j(\pi_*\mathcal{F})^G) \Rightarrow H^{i+j}(X, \mathcal{F})^G$$

*Proof:*

Define the functor  $\Gamma_X^G : \mathbf{QCoh}^G(\mathcal{O}_X) \rightarrow \mathbf{Vect}_k$  by  $\Gamma_X^G(\mathcal{F}) := \Gamma(X, \mathcal{F})^G$ . Note that  $\Gamma_X^G = (-)^G \circ \Gamma_X$  and thus is left-exact. By Maschke's theorem,  $k$  (equipped with trivial action) is a projective  $kG$ -module. This implies the group cohomology  $H_k^1(G, -) = \text{Ext}_{kG}^1(k, -) = 0$ , so  $(-)^G : kG\text{-mod} \rightarrow \mathbf{Vect}_k$  is exact and commutes with cohomology, i.e.  $R^i\Gamma_X^G\mathcal{F} \cong (R^i\Gamma_X\mathcal{F})^G = H^i(X, \mathcal{F})^G$ .

Define another functor  $\pi_*^G : \mathbf{QCoh}^G(\mathcal{O}_X) \rightarrow \mathbf{QCoh}(\mathcal{O}_{X/G})$  by  $\pi_*^G(\mathcal{F}) := (\pi_*\mathcal{F})^G$ . We claim this functor is left-exact. Let  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$  be an exact sequence in  $\mathbf{QCoh}^G(\mathcal{O}_X)$ . Since  $\pi^* \dashv \pi_*$ ,  $\pi_*$  is left-exact, so we have an exact sequence  $0 \rightarrow \pi_*\mathcal{F} \rightarrow \pi_*\mathcal{G} \rightarrow \pi_*\mathcal{H}$ . To check exactness after taking invariants, it suffices to check this on stalks. Since the  $G$ -action is a morphism of sheaves, note that for open  $U \subseteq X/G$  and  $s \in \pi_*\mathcal{F}(U)$ ,  $(g.s)|_V = g.s|_V$  for all  $V \subseteq U$ . Because of this, the  $G$ -action on sheaves extends to a well-defined action on  $(\pi_*\mathcal{F})_x$  for each  $x \in G$  given by  $g \cdot \langle U, s \rangle = \langle U, g.s \rangle$ . It is clear from this definition that

$$(\pi_*^G \mathcal{F})_x = \lim_{\rightarrow U \ni x} (\pi_* \mathcal{F}(U))^G \cong \left( \lim_{\rightarrow U \ni x} \pi_* \mathcal{F}(U) \right)^G = (\pi_* \mathcal{F})_x^G$$

Since  $(-)^G$  is exact on  $kG$ -modules by the discussion above, and since  $0 \rightarrow (\pi_* \mathcal{F})_x \rightarrow (\pi_* \mathcal{G})_x \rightarrow (\pi_* \mathcal{H})_x$  is exact for all  $x \in X/G$ , it follows that  $0 \rightarrow \pi_*^G \mathcal{F} \rightarrow \pi_*^G \mathcal{G} \rightarrow \pi_*^G \mathcal{H}$  is exact on all stalks and thus is exact, therefore  $\pi_*^G$  is left-exact.

We claim  $\pi_*^G$  sends injective objects to  $\Gamma_{X/G}$ -acyclic ones. Since injective objects in  $\mathbf{QCoh}^G(\mathcal{O}_X)$  are flasque, it suffices to show that  $\pi_*^G$  preserves flasqueness. Indeed, we know that  $\pi_*$  preserves flasqueness, so if  $V \subseteq U \subseteq X/G$  and  $\mathcal{F}$  is a flasque  $\mathcal{O}_X$ -module, then there is a surjection  $\Gamma(U, \pi_* \mathcal{F}) \twoheadrightarrow \Gamma(V, \pi_* \mathcal{F})$ . By a similar argument to the paragraph above, taking  $G$ -invariants in  $\mathbf{QCoh}(\mathcal{O}_{X/G})[G]$  is exact, thus  $\Gamma(U, \pi_* \mathcal{F})^G = \Gamma(U, \pi_*^G \mathcal{F}) \twoheadrightarrow \Gamma(V, \pi_* \mathcal{F})^G = \Gamma(V, \pi_*^G \mathcal{F})$ , thus  $\pi_*^G \mathcal{F}$  is flasque. Noting that

$$\Gamma_{X/G} \circ \pi_*^G \mathcal{F} = \Gamma(X/G, (\pi_* \mathcal{F})^G) = \Gamma(X/G, \pi_* \mathcal{F})^G = \Gamma(X, \mathcal{F})^G = \Gamma_X^G(\mathcal{F})$$

It follows from [7, Theorem 5.8.3] that we have a Grothendieck spectral sequence

$$R^i \Gamma_{X/G} \circ R^j \pi_*^G(\mathcal{F}) \Rightarrow R^{i+j} \Gamma_X^G(\mathcal{F})$$

Putting these two facts together and using the fact that sheaf cohomology are the derived functors of the global sections functor, we find

$$H^i(X/G, R^j(\pi_* \mathcal{F})^G) \Rightarrow H^{i+j}(X, \mathcal{F})^G$$

Finally, we want to consider the functors  $R^j(\pi_* \mathcal{F})^G$ . Note that  $\pi_*^G = (-)^G \circ \pi_*$ , where  $(-)^G : \mathbf{QCoh}(\mathcal{O}_{X/G})[G] \rightarrow \mathbf{QCoh}(\mathcal{O}_{X/G})$  described above the lemma. By a similar argument to showing  $\pi_*^G$  is left-exact, it can be shown that  $(-)^G$  is exact and therefore commutes with cohomology, so that  $R^i \pi_*^G(\mathcal{F}) \cong (R^i \pi_*(\mathcal{F}))^G$ . Thus we get the spectral sequence

$$H^i(X/G, (R^j \pi_* \mathcal{F})^G) \Rightarrow H^{i+j}(X, \mathcal{F})^G$$

■

### Corollary 2.1

If  $\mathcal{F} \in \mathbf{Ob}(\mathbf{Coh}^G(\mathcal{O}_X))$ , then  $H^i(X/G, (\pi_* \mathcal{F})^G) \cong H^i(X, \mathcal{F})^G$  for all  $i \geq 0$ .

*Proof:*

By [8], since  $\mathcal{F}$  is a coherent  $\mathcal{O}_X$ -module,  $\pi : X \rightarrow X/G$  a finite map of schemes over  $k$ , and  $X/G$  locally Noetherian, we have  $R^i \pi_*(\mathcal{F}) = 0$  for  $i > 0$ . Thus the spectral sequence of Theorem 2.1 collapses, yielding the desired isomorphism.

■

In particular, since the sheaf of differentials is coherent, we find

$$H^p(X/G, (\pi_* \Omega_X^q)^G) \cong H^p(X, \Omega_X^q)^G$$

We conclude this section with a result regarding the hypercohomology of chain complexes of  $G$ -equivariant coherent sheaves over a  $k$ -scheme  $X$ . While these results are not immediately useful for our purposes, we include them for sake of completion.

### Corollary 2.2

Let  $\mathcal{F}^\bullet$  be a chain complex in  $\mathbf{Coh}^G(\mathcal{O}_X)$ . Then for all  $n \in \mathbb{Z}_{\geq 0}$

$$\mathbb{H}^n(X/G, (\pi_* \mathcal{F}^\bullet)^G) \cong \mathbb{H}^n(X, \mathcal{F})^G$$

*Proof:*

We have the hypercohomology spectral sequence

$$H^p(X/G, H^q((\pi_* \mathcal{F}^\bullet)^G)) \Rightarrow \mathbb{H}^{p+q}(X/G, (\pi_* \mathcal{F}^\bullet)^G)$$

Because  $\pi : X \rightarrow X/G$  is a proper map ( $X$  being compact), for any coherent  $\mathcal{O}_X$ -module  $\mathcal{G}$ ,  $R^i \pi_*(\mathcal{G}) = 0$  for  $i > 0$  and  $\pi_* \mathcal{G}$  is coherent, thus  $\pi_* : \mathbf{Coh}(\mathcal{O}_X) \rightarrow \mathbf{Coh}(\mathcal{O}_{X/G})$  is an exact functor. Since the characteristic of  $k$  is zero,  $(-)^G$  is also exact. Thus because  $\mathcal{F}^\bullet$  is a chain of coherent  $\mathcal{O}_X$  modules, there are natural isomorphisms

$$H^p((\pi_* \mathcal{F}^\bullet)^G) \cong H^p(\pi_* \mathcal{F}^\bullet)^G \cong (\pi_* H^p(\mathcal{F}^\bullet))^G$$

And so we have a natural isomorphism

$$H^p(X/G, H^q((\pi_* \mathcal{F}^\bullet)^G)) \cong H^p(X/G, (\pi_* H^q(\mathcal{F}^\bullet))^G)$$

Since  $\mathbf{Coh}(\mathcal{O}_X)$  is closed under quotients,  $H^q(\mathcal{F}^\bullet) \in \mathbf{Ob}(\mathbf{Coh}(\mathcal{O}_X))$ , so by Theorem 2.1

$$H^p(X/G, (\pi_* H^q(\mathcal{F}^\bullet))^G) \cong H^p(X, H^q(\mathcal{F}^\bullet))^G$$

On the other hand, we have the hypercohomology spectral sequence

$$H^p(X, H^q(\mathcal{F}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet)$$

Because  $(-)^G : kG\text{-mod} \rightarrow \mathbf{Vect}_k$  is exact and because the differentials in  $H^p(X, H^q(\mathcal{F}^\bullet))$  are  $G$ -equivariant, we obtain another spectral sequence

$$H^p(X, H^q(\mathcal{F}^\bullet))^G \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet)^G$$

It can be checked that the isomorphisms  $H^p(X/G, (\pi_* \mathcal{F}^\bullet)^G) \cong H^p(X, H^q(\mathcal{F}^\bullet))^G$  are ones of spectral sequences, so by [7, Comparison Theorem 5.2.12]

$$\mathbb{H}^n(X/G, (\pi_* \mathcal{F}^\bullet)^G) \cong \mathbb{H}^n(X, \mathcal{F}^\bullet)^G$$

■

In the next section, we will apply these results to the toy example discussed in “Motivation and Setup.”

### 3. Hodge Structures of Certain Godeaux Surfaces

We now turn our attention to the aforementioned toy example. Since we are working over  $\mathbb{C}$ , we have  $(\pi_* \Omega_X^p)^G \cong \Omega_{X/G}^p$ , thus  $H^q(X/G, (\pi_* \Omega_X^p)^G) \cong H^q(X/G, \Omega_{X/G}^p)$ .

#### Proposition 3.3

*Let  $X$  be a hypersurface as described in “Motivation and Setup.” Then the Hodge diamond of the quotient manifold  $X/G$  is given by*

$$\begin{array}{ccccc}
& & 1 & & \\
& & 0 & & 0 \\
0 & & 9 & & 0 \\
& & 0 & & 0 \\
& & 1 & & 
\end{array}$$

*Proof:*

We make use of the Hodge decomposition theorem, which states that for a complex projective variety  $Y$  the singular cohomology decomposes into the following short exact sequence as

$$H^d(Y; \mathbb{C}) \cong \bigoplus_{p+q=d} H^p(Y, \Omega_Y^q)$$

In particular, the rank of  $H^0(Y; \mathbb{C}) \cong H^0(Y, \Omega_Y^0)$  is the number of connected components. Since  $H^0(X, \Omega_X^0) \cong k$  by Proposition 2,  $X$  has a single connected component, and since the quotient map  $X \rightarrow X/G$  is surjective,  $X/G$  also has a single connected component, thus  $H^0(X/G, \Omega_{X/G}^0) \cong k$ .

Similarly, we know that every complex manifold is orientable and thus has top cohomology isomorphic to  $k$ . Since the  $\mathbb{C}$ -dimension of  $X/G$  is  $\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} G = 2 - 0 = 2$ , it has  $\mathbb{R}$ -dimension 4, thus  $H^4(X/G; \mathbb{C}) \cong H^2(X/G, \Omega_{X/G}^2) \cong \mathbb{C}$ .

From the comments after Corollary 2.1,  $H^p(X/G, \Omega_{X/G}^q) \cong H^p(X, \Omega_X^q)^G$ . From this, it immediately follows that  $h_{X/G}^{10} = h_{X/G}^{01} = h_{X/G}^{12} = h_{X/G}^{21} = 0$ .

The computation of  $H^2(X, \Omega_X^0)^G$  follows from the natural isomorphism

$$H^2(X, \Omega_X^0) \cong H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-5)) = \left( \frac{1}{xyz} \mathbb{C} \left[ \frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right] \right)_{-5}$$

In particular, this isomorphism commutes with the group action isomorphisms. Tracing definitions, these induce the obvious action on the explicit form written to the right, which has basis  $\left\{ \frac{1}{w^2xyz}, \frac{1}{wx^2yz}, \frac{1}{wxy^2z}, \frac{1}{wxyz^2} \right\}$ . Computing the action on each basis element, we find that the  $\mathbb{Z}/5\mathbb{Z}$ -action acts on each basis element by multiplication by a nontrivial fifth root of unity and thus decomposes the short exact sequence as a  $\mathbb{C}[G]$ -module into a direct sum of nontrivial  $\mathbb{Z}/5\mathbb{Z}$ -irreps. Thus  $H^2(X/G, \Omega_{X/G}^0) \cong H^2(X, \Omega_X^0)^G = 0$ . By Hodge symmetry, this means  $h_{X/G}^{20} = h_{X/G}^{02} = 0$ .

It remains to compute the  $k$ -dimension of  $H^1(X/G, \Omega_{X/G}^1) \cong H^1(X, \Omega_X^1)^G$ . Since  $(-)^G : kG\text{-mod} \rightarrow \mathbf{Vect}_k$  is exact, we have from proof of Proposition 1.2 that

$$h_{X/G}^{11} = \dim_k H^1(\mathbb{P}_k^3, \Omega_{\mathbb{P}}^1)^G + \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-10))^G - 4 \dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-6))^G$$

From Appendix,  $\dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-6))^G = 2$  and  $\dim_k H^3(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}(-10))^G = 16$ . From the proof of Proposition 1, there are natural isomorphisms

$$(k[w, x, y, z])_0 = \Gamma(\mathbb{P}_k^3, \mathcal{O}_{\mathbb{P}}) \cong H^1(\mathbb{P}_k^3, i_* i^* \Omega_{\mathbb{P}}^1) \cong H^1(\mathbb{P}_k^3, \Omega_{\mathbb{P}}^1)$$

Since  $\mathbb{Z}/5\mathbb{Z}$  acts trivially on  $(k[w, x, y, z])_0$ , it follows that  $\dim_k H^1(\mathbb{P}_k^3, \Omega_{\mathbb{P}}^1)^G = 1$ , thus

$$h_{X/G}^{11} = 1 + 16 - 2(4) = 17 - 8 = 9$$

■

In particular, for each  $n$  we see that the Hodge filtration of  $H^n(X/G, \mathbb{C})$  is zero for  $i > n$  and the entire cohomology for  $i \leq n$ . Now let  $W$  be a deformation of some  $X$ . By the “Motivation and Setup” section, each fiber of  $W/G$  is diffeomorphic to  $X/G$ , thus for each  $n$  and fiber  $Y_t$  we have  $H^n(X/G, \mathbb{C}) \cong H^n(Y_t, \mathbb{C})$ . It is immediately obvious that this extends to an isomorphism of Hodge structures. While we have not yet verified that the mixed Hodge structure on the  $\mathbb{Q}[\pi_1(X/G)]/\mathcal{J}^s$  are isomorphic for all fibers, this example proves promising for providing explicit constructions of deformations satisfying the conditions for the problem described in the “Motivation and Setup” section.

In the next section, we recall a few categorical constructions, and provide definitions for Hodge-refined rings and modules.

## 4. Hodge-Refined Rings and Modules

**Definition 4.1** (monad)

A monad on a category  $\mathcal{C}$  is the data of an endofunctor  $T : \mathcal{C} \rightarrow \mathcal{C}$ , a “unit” natural transformation  $\eta : \text{Id}_{\mathcal{C}} \Rightarrow T$ , and a “multiplication” natural transformation  $\mu : T^2 \Rightarrow T$  such that the following diagrams commute

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \mu_{T(-)} \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{T\eta} & T^2 \\ \eta_{T(-)} \downarrow & \searrow & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

//

Let  $R$  be a ring. Then  $R \otimes (-) : \mathbf{Ab} \rightarrow \mathbf{Ab}$  is a monad, the unit given by  $\eta_X : x \mapsto 1 \otimes x$  and the multiplication by  $\mu_X : r \otimes s \otimes x \mapsto rs \otimes x$ . Here  $X$  is an arbitrary abelian group,  $x \in X$ , and  $r, s \in R$ . We could hope that given a ring  $R$  equipped with some mixed Hodge structure, these same maps would be ones of mixed Hodge structures. It turns out, this will only be the case if the ring multiplication satisfies certain compatibility conditions with respect to the Hodge and weight filtrations.

**Lemma 4.1**

Let  $(R, W_{\bullet}, F^{\bullet})$  be a mixed Hodge structure. Suppose  $R$  is equipped with a ring structure. Then  $R \otimes (-) : \mathbf{MHS} \rightarrow \mathbf{MHS}$  is a monad in the way described above iff the following conditions are met:

- (1)  $1 \in W_{\ell} R$  for some  $\ell \leq 0$
- (2)  $1 \in F^{\ell} R$  for some  $\ell \geq 0$
- (3) For any  $u, v \in \mathbb{Z}$ ,  $(W_u R)(W_v R) \subseteq W_t(V)$  for some  $t \leq u + v$
- (4) For any  $u, v \in \mathbb{Z}$ ,  $(F^u R)(F^v R) \subseteq F^t(V)$  for some  $t \geq u + v$

*Proof:*

Suppose  $R \otimes (-) : \mathbf{MHS} \rightarrow \mathbf{MHS}$  is a monad. Then for all  $X \in \mathbf{Ob}(\mathbf{MHS})$ , the following unit map is one of mixed Hodge structures:

$$\eta_X : X \rightarrow R \otimes X$$

$$1 \mapsto 1 \otimes x$$

In particular, for any  $p$  we have  $\eta_X(F^p X) \subseteq F^p(R \otimes X)$  and  $\eta_X(W_p X) \subseteq W_p(R \otimes X)$ . Fix some  $X \in \text{Ob}(\text{MHS})$ ,  $p \in \mathbb{Z}$ , and  $x \in X$  such that  $x \in W_p X \setminus W_{p-1} X$ . Note that this latter condition is possible since  $W_\bullet$  is increasing. These constraints also imply that if  $x \in W_k X$ , then  $k \geq p$ . Since  $1 \otimes x \in W_p(R \otimes X) = \bigoplus_{\ell+k=p} W_\ell R \otimes W_k X$ ,  $1 \in W_\ell R$  and  $x \in W_k X$  for some integers  $\ell, k \in \mathbb{Z}$  such that  $\ell + k = p$ . But  $k \geq p$ , thus

$$p = \ell + k \geq \ell + p \implies 0 \leq \ell$$

An analogous argument shows that  $1 \in F^\ell R$  for some  $\ell \geq 0$  upon noting that  $F^\bullet$  is a decreasing filtration.

Turning to the multiplication map, for all  $X \in \text{Ob}(\text{MHS})$ , we have  $\mu_X(F^p(R \otimes R \otimes X)) \subseteq F^p(R \otimes X)$  and  $\mu_X(W_p(R \otimes R \otimes X)) \subseteq W_p(R \otimes X)$  since  $\mu$  is an MHS map. Take  $X, p \in \mathbb{Z}$ , and  $x \in X$  as before. Let  $r, s \in R$  be arbitrary. We have  $r \in W_u R$  and  $s \in W_v R$  for some  $u, v \in \mathbb{Z}$ . Then  $r \otimes s \otimes x \in W_{u+v+p}(R \otimes R \otimes X)$ , so  $rs \otimes x \in W_{u+v+p}(R \otimes X) = \bigoplus_{t+k=u+v+p} W_t R \otimes W_k X$ . By the same argument as before,  $k \geq p$ , therefore

$$t + k = u + v + p \geq t + p \implies t \leq u + v$$

Analogous reasoning shows that if  $r \in F^u R, s \in F^v R$ , then  $rs \in F^t R$  for some  $t \geq u + v$ , using that  $F^\bullet$  is decreasing.

Since the unit  $\eta$  and multiplication  $\mu$  are setwise the same as those for  $R \otimes (-) : \mathbf{Ab} \rightarrow \mathbf{Ab}$ , the required diagrams commute, so it suffices to show that they are maps of mixed Hodge structures.

Now suppose  $R$  satisfies properties (1) through (4). Take any  $X \in \text{Ob}(\text{MHS})$ . Each element  $x \in X$  is in  $W_p X$  for some  $p$  and each  $W_p X$  contains some element of  $x$ , so it suffices to show that  $1 \otimes x \in W_p(R \otimes X)$ . Since  $1 \in W_\ell R$  for  $\ell \leq 0$  and  $W_\bullet$  is increasing, it follows that  $x \in W_{p-\ell} X \supseteq W_p X$ , thus  $1 \otimes x \in W_\ell R \otimes W_{p-\ell} X \subseteq \bigoplus_{\ell+k=p} W_\ell R \otimes W_k X = W_p(R \otimes X)$ . The fact that  $\eta_X(F^p X) \subseteq F^p(R \otimes X)$  for arbitrary  $X$  and  $p$  follows similarly.

Applying similar logic as the preceding paragraph and using the fact that simple tensors generate the tensor product of two spaces, it suffices to show that for any  $r, s \in R$  with  $r \in W_u R, s \in W_v X$  and  $x \in X \in \text{Ob}(\text{MHS})$  with  $x \in W_p X$  we have  $rs \otimes x \in W_{u+v+p}(R \otimes X)$ . We know  $rs \in W_t R$  for some  $t \leq u + v$  and that  $W_\bullet X$  is increasing, thus  $x \in W_{p+u+v-t} X \supseteq W_p X$ , and we see that  $rs \otimes x \in W_t R \otimes W_{p+u+v-t} X \subseteq \bigoplus_{t+k=u+v+p} W_t R \otimes W_k X = W_{u+v+p}(R \otimes X)$ . A similar argument also shows that  $\mu_X(F^{u+v+p}(R \otimes R \otimes X)) \subseteq F^{u+v+p}(R \otimes X)$ . Since we are free to choose  $u, v, p \in \mathbb{Z}$ , it follows that  $\mu_X$  and  $\eta_X$  are maps of mixed Hodge structures for all  $X \in \text{Ob}(\text{MHS})$ , thus  $R \otimes (-) : \text{MHS} \rightarrow \text{MHS}$  is a monad. ■

#### Definition 4.2 (Hodge-refined ring)

A Hodge-refined ring is a ring  $R$  equipped with a mixed Hodge structure satisfying the conditions of Lemma 3.1.

//

#### Definition 4.3 (T-algebra)

Let  $(T, \eta, \mu)$  be a monad over  $\mathcal{C}$ . A  $T$ -algebra is a pair  $(X, h)$  with  $X \in \text{Ob}(\mathcal{C})$  and  $h : TX \rightarrow X$  such that the following diagrams commute

$$\begin{array}{ccc}
T^2X & \xrightarrow{\mu_X} & TX \\
Th \downarrow & & \downarrow h \\
TX & \xrightarrow{h} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & TX \\
\parallel & & \downarrow h \\
& & X
\end{array}$$

//

**Definition 4.4** ( $R_{\text{MHS}}$ -module)

Let  $R \in \text{Ob}(\text{MHS})$  be as in Lemma 1. Then an  $R_{\text{MHS}}$ -module is an  $(R \otimes (-))$ -algebra. We will also refer to such an object as a Hodge-refined module.

//

**Lemma 4.2**

The forgetful functor  $U : \text{MHS} \rightarrow \text{Ab}$  induces a forgetful functor  $R_{\text{MHS}}\text{-mod} \rightarrow R\text{-mod}$ .

*Proof:*

The forgetful functor  $U$  sends an  $R_{\text{MHS}}$ -module  $(X, h)$  to the pair  $(X, h)$ , where we view  $X \in \text{Ob}(\text{Ab})$  and  $R \otimes X \xrightarrow{h} X \in \text{Mor}(\text{Ab})$ . By functoriality of  $U$ , the diagrams of an  $(R \otimes (-))$ -algebra are still satisfied (viewed in  $\text{Ab}$ ), thus  $U(X, h)$  is an  $(R \otimes (-))$ -algebra in  $\text{Ab}$ . It is well-known that  $(R \otimes (-))\text{-alg} = R\text{-mod}$ , which proves the claim. ■

We can then ask under what conditions an  $R$ -module equipped with a MHS is an  $R_{\text{MHS}}$ -module. This is answered in the following lemma. The proof is almost identical to that of Lemma 4.1 and so is omitted.

**Lemma 4.2**

Let  $R$  be a Hodge-refined ring and  $X$  an  $R$ -module equipped with a mixed Hodge structure. Then the  $R$ -action on  $X$  makes  $X$  is an  $R_{\text{MHS}}$ -module iff the following are satisfied

- (1) For any  $u, v \in \mathbb{Z}$ ,  $(W_u R)(W_v X) \subseteq W_t(X)$  for some  $t \leq u + v$
- (2) For any  $u, v \in \mathbb{Z}$ ,  $(F^u R)(F^v X) \subseteq F^t(X)$  for some  $t \geq u + v$ .

In the next section, we discuss how we expect to use the tools we have developed.

## 5. Future Work

We have two main conjectures.

**Conjecture 5.1**

The truncated group rings  $\mathbb{Q}[\pi_1(Y)]/\mathcal{J}^s$  are Hodge-refined rings for all  $s \geq 2$ .

**Conjecture 2**

Let  $W/G$  be a deformation of the toy example  $X/G$  in the manner described earlier. Then the mixed Hodge structure on the truncated group rings of the fundamental groups of the fibers  $Y_t$  of  $W/G$  is independent of the choice of  $t$ .

If these two conjectures turn out to be true, we plan to mimic the construction of Whitehead torsion to define our invariant. We will first turn our attention to the toy example  $X/G$  with

deformation  $W/G$ , and determine whether there is a sensible way in which the truncated group ring of the fundamental group acts on the cohomology of  $W$ , i.e. in a way which makes the cohomology and resolution defining the cohomology of  $W$  into a Hodge-refined  $\mathbb{Q}[G]/\mathcal{J}^s$ -module. Given an  $h$ -cobordism  $(M, N, W', i, j)$ , the Whitehead torsion is defined using a nullhomotopy of the relative chain complex  $C_\bullet(\tilde{M}, \tilde{W}')$ , where  $\tilde{M}, \tilde{W}'$  are the universal covers of  $M, W'$  (respectively). We will attempt to replicate this construction by first finding an acyclic complex relating  $X$  and  $W$ , and then requiring the nullhomotopy to be a map of Hodge-refined modules. Obviously, these next few steps are purely conjectural, and there will likely be many roadblocks to proceeding in this manner.

Although the Hodge structures on the fibers of  $W/G$  are all isomorphic for our toy example, the Hodge structures of the fibers of  $W$  may not be isomorphic. Another approach to defining Hodge-refined Whitehead torsion would be to utilize this fact, however, this would likely deviate significantly from the definition of Whitehead torsion, and thus there is not as clear of a path forward.

It is worth noting that the toy example we have constructed may prove to be trivial due to dimension reasons of due to the small size of  $\mathbb{Z}/5\mathbb{Z}$  and thus provide no insights. In this case we would need to work to construct a more complicated example.

Finally, the definitions of Hodge-refined rings and modules are novel and thus largely unexplored. Future research projects could explore properties of  $R_{\text{MHS}}\text{-mod}$ . In particular, the functor of Lemma 4.2 can be shown to be exact and thus induces a map between the  $K$ -theory of finitely-generated  $R_{\text{MHS}}$ -modules and that of finitely-generated  $R$ -modules.

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## Appendix

Consider the  $k$ -vector space

$$\left( \frac{1}{wxyz} k \left[ \frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right] \right)_{-d}$$

This has a basis consisting of those elements of the form

$$\frac{1}{w^{e_0} x^{e_1} y^{e_2} z^{e_3}} \text{ such that } e_i \geq 1 \text{ for all } i \text{ and } e_0 + e_1 + e_2 + e_3 = d$$

It is clear the basis elements are in bijective correspondence with 4-digit base  $d$  numbers containing no zeros and whose digits sum to  $d$ . When  $d \leq 10$ , we can check how many of these there are with brute force. I used the `count_numbers` function in the Python code below to compute how many of such numbers there are for  $d = 6, 10$ .

Next, the group action on  $\frac{1}{w}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$  yields  $\frac{1}{w}, \frac{1}{\zeta x}, \frac{1}{\zeta^2 y}, \frac{1}{\zeta^3 z}$  (resp.). Since each basis is a monomial in these variables and the group action scales by a root of unity, the  $kG$ -representation decomposes into exact sequence according to this basis, and the  $G$ -invariants will be spanned by those basis vectors upon which the group acts trivially. Since  $G$  is cyclic, this

will hold iff  $e_1 + 2e_2 + 3e_3 \equiv 0 \pmod{5}$ . I used the `count_numbers_weighted` function in the code below to compute how many of these basis vectors there are.

```
def weight_sum(number):
    digits = [int(d) for d in str(number)]

    total_sum = 0

    for i in range(0, len(digits)):
        total_sum += i*digits[i]

    return total_sum

def add_dig(number):
    split = [int(d) for d in str(number)]

    j = 0
    for i in split:
        j += i

    return j

def count_numbers(num_dig, target_sum):
    counter = 0

    for i in range(10**(num_dig-1), 10**num_dig-1):
        if add_dig(i) == target_sum and not 0 in [int(d) for d in str(i)]:
            counter += 1

    return counter

def count_numbers_weighted(num_dig, target_sum):
    counter = 0

    for i in range(10**(num_dig-1), 10**num_dig-1):
        if add_dig(i) == target_sum and not 0 in [int(d) for d in str(i)] and
weight_sum(i) % 5 == 0:
            counter += 1

    return counter

print(count_numbers(4,10))
print(count_numbers(4,6))

print(count_numbers_weighted(4,10))
print(count_numbers_weighted(4,6))
```

This gave outputs of 84, 10, 16, and 2 (resp.).

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