

Negligible Group Cohomology Over Fields of Characteristic p , Eventually Negligible Classes in Integral and Mod n Group Cohomology, and Negligible Properties of the Bockstein Homomorphism

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Abstract

Let F be a field, K/F a field extension with $\Gamma := \text{Gal}(K_{\text{sep}}/K)$, G a finite group, $j : \Gamma \rightarrow G$ a continuous group homomorphism, and M an abelian group. Then there is an induced map $j^* : H^n(G; M) \rightarrow H^n(K; M)$. We say an element $u \in H^n(G; M)$ is negligible over F if $j^*(u) = 0$ for all such j and field extensions K/F . The space of negligible elements forms a subgroup $H^n(G; M)_{\text{neg}} \subseteq H^n(G; M)$ containing information about group invariants and Galois embeddings over F . Given a unital commutative ring R , we can form the graded anticommutative cohomology ring $\mathcal{H}(G; R) := \bigoplus_{n \in \mathbb{N}} H^n(G; R)$. We say an element $u \in H^n(G; R)$ is eventually negligible over F if there exists some $k \in \mathbb{N}$ such that, with the respect to the ring multiplication on $\mathcal{H}(G; R)$, $u^k \in H^{kn}(G; R)_{\text{neg}}$. In this project, we compute the second degree negligible cohomology of cyclic p -groups with coefficients in the Prüfer Group over fields of characteristic zero with certain roots of unity assumptions. We prove several results concerning negligible cohomology over fields of characteristic p for arbitrary groups; in particular, we provide a complete description of negligible cohomology for a p -group G over fields of characteristic p with coefficients in finitely-generated abelian groups and direct limits of these. We show that every element of degree greater than or equal to zero is eventually negligible over a field with sufficiently many roots of unity in $\mathcal{H}(G; \mathbb{Z}/n\mathbb{Z})$ and $\mathcal{H}(G; \mathbb{Z})$ for $n \in \mathbb{N}$. Finally, we explore properties of the Bockstein homomorphisms in identifying negligible classes.

Notation

The degree $n \in \mathbb{N}$ group cohomology of a group G with coefficients M , where M is a G -module, is denoted $H^n(G; M)$. We often write $H^n(G; M)$ without specifying n , G , and M , so unless otherwise stated, n , G , and M will always be the aforementioned structures. Throughout this paper we will always assume the coefficients in our group cohomology to have trivial G -action, thus, the coefficients M are abelian groups.

Throughout this paper, we will use the following notations. For a group $\mathbb{Z}/m\mathbb{Z}$, $H^n(G; m) := H^n(G; \mathbb{Z}/m\mathbb{Z})$ and $H^n(m; M) := H^n(\mathbb{Z}/m\mathbb{Z}; M)$.

For $p \in \mathbb{N}$ a prime, $\mathbb{Z}/p^\infty\mathbb{Z} := \text{colim}_i \mathbb{Z}/p^i\mathbb{Z}$ denotes the Prüfer group.

Any result with “(N)” next to it is novel.

Background and Motivation

Galois cohomology modules are of vital importance within field theory and number theory, as they provide extremely powerful isomorphism invariants: powerful in the sense that the structure of the cohomologies provides information about a variety of interactions between the objects involved in

the computations. For example, the tools provided from Galois cohomology yield a simple proof that given a Galois extension L/K , the Brauer group $\text{Br}(L/K) \subseteq \text{Br}(K)$ [Weib, p. 214]. Other examples are given in Vista 6.11.19 [Weib, p. 214-215].

It is a common trope throughout mathematics to partially solve difficult problems by “approximating” with easier problems. Such is the motivation for negligible group cohomology; suppose we are given a field F , K/F a field extension with absolute Galois group Γ_K , a finite group G , and a continuous group homomorphism $j : \Gamma_K \rightarrow G$. Because G is finite, the continuous group homomorphisms are precisely the group homomorphisms. By definition, the Galois cohomology $H^n(K; M)$ is the group cohomology $H^n(\Gamma_K; M)$, so by the contravariant functoriality of group cohomology, we can consider the induced maps $j^* : H^n(G; M) \rightarrow H^n(K; M)$. In the case that j is surjective, we would hope that j^* is injective, however, this is oftentimes not the case. The negligible cohomology $H^d(G; M)_{\text{neg}}$ consists of the classes $u \in H^n(G; M)$ such that $j^*(u) = 0$ for all maps $j : \Gamma_K \rightarrow G$ and all field extensions K/F . In this sense, the computations of $H^n(G; M)$ may be used to “approximate” $H^n(F; M)$, with the “size” of the negligible cohomology submodule giving an idea of how accurate such approximations are.

A list of other mathematical applications of negligible classes is found in Example 1.1 [Gher].

In this paper, we first compute $H^2(p^k; p^\infty)_{\text{neg}}$ for fields of characteristic zero with exactly the μ_{p^n} roots of unity, where $p \in \mathbb{N}$ is prime. Next, we provide a partial description of the negligible cohomology of an arbitrary group G with coefficients in $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}$ over fields of prime characteristic p . We then show that over a field with sufficiently many roots of unity, there exists some $D \in \mathbb{N}$ such that for $n > D$ and coefficients in $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, the n -th degree negligible cohomology is precisely the group cohomology. Finally, we explore the properties of the Bockstein homomorphism $H^n(G; p) \rightarrow H^{n+1}(G; p^2)$ in identifying negligible classes.

1. Computation of $H^2(p^k; p^\infty)_{\text{neg}}$

Proposition 1.1

Let $n, k, \ell, p \in \mathbb{N}$ with p prime and $n \geq \ell$. We compute $H^2(p^k; p^\ell)_{\text{neg}}$ over a field F of characteristic not dividing p^ℓ and with $\mu(F) \cong \mu_{p^n}$.

Proof:

Since $\mathbb{Z}/m\mathbb{Z} \cong \mu_m$ for all $m \in \mathbb{N}$, We have from Corollary 3.2 [Gher] that $H^2(p^k; p^\ell)_{\text{neg}}$ over a field F is the image of the morphism δ in the exact sequence below:

$$0 \rightarrow H^1(p^k; p^\ell) \rightarrow H^1(p^k; p^n) \xrightarrow{q_*} H^1(p^k; p^{n-\ell}) \xrightarrow{\delta} H^2(p^k; p^\ell)$$

By the first isomorphism theorem and exactness of the sequence above:

$$\text{im } \delta \cong H^1(p^k; p^{n-\ell}) / \ker \delta \cong H^1(p^k; p^{n-\ell}) / \text{im } q_* \cong \text{coker } q_*$$

From this, we see that $H^2(p^k; p^\ell)_{\text{neg}}$ is the unique object making the following sequence exact:

$$0 \rightarrow H^1(p^k; p^\ell) \cong \ker q_* \rightarrow H^1(p^k; p^n) \xrightarrow{q_*} H^1(p^k; p^{n-\ell}) \rightarrow H^2(p^k; p^\ell)_{\text{neg}} \rightarrow 0$$

Since we are assuming trivial action, $H^1(G; M) \cong \text{Hom}_{\text{Grp}}(G, M)$, so the above sequence becomes:

$$0 \rightarrow \text{Hom}(p^k, p^\ell) \rightarrow \text{Hom}(p^k, p^n) \xrightarrow{q_*} \text{Hom}(p^k, p^{n-\ell}) \rightarrow H^2(p^k; p^\ell)_{\text{neg}} \rightarrow 0$$

From group theory, $\text{Hom}(p^r, p^s) \cong \mathbb{Z}/p^m\mathbb{Z}$, where $m = \min\{r, s\}$. With this fact in mind, we split into eight cases:

$$\begin{aligned}
& 0 \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^{n-\ell}\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^{n-\ell}\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^\ell\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^\ell\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \mathbb{Z}/p^{n-\ell}\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^\ell\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0 \\
& 0 \rightarrow \mathbb{Z}/p^\ell\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^{n-\ell}\mathbb{Z} \rightarrow \text{coker } q^* \rightarrow 0
\end{aligned}$$

We can quickly compute cokernels and thus negligible cohomologies for each case; let $r, s, t \in \mathbb{N}$ and suppose we have the following exact sequence:

$$0 \rightarrow \mathbb{Z}/p^r\mathbb{Z} \cong \ker \phi \rightarrow \mathbb{Z}/p^s\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/p^t\mathbb{Z} \rightarrow \text{coker } \phi \rightarrow 0$$

By the first isomorphism theorem and abusing notation:

$$\text{im } \phi \cong (\mathbb{Z}/p^s\mathbb{Z}) / \ker \phi \cong (\mathbb{Z}/p^s\mathbb{Z}) / (\mathbb{Z}/p^r\mathbb{Z}) \cong \mathbb{Z}/p^{s-r}\mathbb{Z}$$

Thus:

$$\text{coker } \phi \cong (\mathbb{Z}/p^t\mathbb{Z}) / \text{im } \phi \cong (\mathbb{Z}/p^t\mathbb{Z}) / (\mathbb{Z}/p^{s-r}\mathbb{Z}) \cong \mathbb{Z}/p^{r-s+t}\mathbb{Z}$$

Using the values of r, s, t from the case-by-case sequences above, we obtain the computation for $H^2(p^k; p^\ell)_{\text{neg}, F}$, where the characteristic of F does not divide p^ℓ and $\mu(F) \cong \mu_{p^n}$. ■

For the computation of $H^2(p^k; p^\infty)_{\text{neg}}$, we need the following lemma.

Lemma 1.1

Let $\{M_i : \varphi_{ij} : M_i \rightarrow M_j\}$ be a filtered system with colimit M . Then $M = 0$ iff the dotted lines in the diagram below are zero for all i, j and for all objects A :

$$\begin{array}{ccc}
& & A \\
& \nearrow \text{dotted} & \nwarrow \text{dotted} \\
M_i & \xrightarrow{\varphi_{ij}} & M_j
\end{array}$$

Suppose the only morphisms shown in the diagram are the zero morphisms. Then the only morphisms $M_i \rightarrow M$ are the zero morphisms, so by the universal property of the colimit, $M = 0$.

On the other hand, suppose there exist morphisms $\psi_i : M_i \rightarrow A$ making the diagram commute for some module A such that there exists an i with $\psi_i \neq 0$. Then by the universal property of the colimit, we may factor ψ_i through M . However, if $M = 0$, then the only such map out of M is the

zero map, and since composition with zero is zero, we have that $\psi_i = 0 \circ \iota_i = 0$ for $\iota_i : M_i \rightarrow M$ the canonical maps, thus $\psi_i = 0$ for all i , a contradiction. It follows that $M \neq 0$. ■

Proposition 1.2 (N)

Let $k, p \in \mathbb{N}$ with p prime. Then for any field F satisfying $\text{char}(F) = 0$ and $\mu(F) \cong \mu_{p^\infty}$, we have that $H^2(p^k; p^\infty)_{\text{neg}} = 0$.

Proof:

Let $\iota_i : \mathbb{Z}/p^i\mathbb{Z} \rightarrow \mathbb{Z}/p^{i+1}\mathbb{Z}$ be defined by $\iota_i(1) = p$. Then the system $\{\mathbb{Z}/p^i\mathbb{Z}, \iota_i\}$ has colimit $\mathbb{Z}/p^\infty\mathbb{Z}$. Using basic facts about directed colimits and the definition of negligible group cohomology:

$$H^2(p^k; p^\infty)_{\text{neg}} \cong H^2(p^k; \text{colim}_\ell \mathbb{Z}/p^\ell\mathbb{Z})_{\text{neg}} \cong \text{colim}_\ell H^2(p^k; p^\ell)_{\text{neg}}$$

Using Computation 1 and that F has exactly the p^n -th roots of unity for all $n \in \mathbb{N}$, it follows that when $\ell > k$, $H^2(p^k; p^\ell)_{\text{neg}} \cong \mathbb{Z}/p^k\mathbb{Z}$. However, we also note that the induced maps $\iota_{i,*} :$

$H^2(p^k; p^\ell)_{\text{neg}} \rightarrow H^2(p^k; p^{\ell+1})_{\text{neg}}$ are still “multiplication by p ” (accounting for the appropriate equivalence classes), we have the following diagram for $\ell \gg k$ with which we take a colimit over:

$$\dots \xrightarrow{p} \mathbb{Z}/p^k\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^k\mathbb{Z} \xrightarrow{p} \dots$$

But now any map out of one of the $H^2(p^k; p^\ell)_{\text{neg}}$ making the diagram commute must be zero since we eventually reach the diagram above and since multiplication by p^k is the zero map in $\mathbb{Z}/p^k\mathbb{Z}$. Expressing visually, we have that the direct limit we are computing, call it $\tilde{M}$, must satisfy the following diagram for all $\ell \gg k$:

$$\begin{array}{ccccc} & & \tilde{M} & & \\ & \nearrow & \uparrow & \nwarrow & \\ \mathbb{Z}/p^k\mathbb{Z} & \xleftarrow{p^k} & \mathbb{Z}/p^k\mathbb{Z} & \xleftarrow{\quad} & H^2(p^k; p^\ell)_{\text{neg}} \end{array}$$

But from this we see that the only map $H^2(p^k; p^\ell)_{\text{neg}}$ satisfying the diagram is the zero diagram for any ℓ since this diagram holds for all sufficiently high ℓ . It follows from Lemma 1.1 that $H^2(p^k; p^\infty)_{\text{neg}} = 0$. ■

2. Fields of Characteristic p

Proposition 2.1

Let F be a field of (prime) characteristic p . Then for all extensions K/F , the Galois cohomology $H^n(K; p) = 0$ for all $n > 1$.

Proof:

Let K be as defined in the context of the problem. Take $b \in K_{\text{sep}}$ and consider the polynomial $f_b(x) := x^p - x + b \in K_{\text{sep}}[x]$. Because K_{sep}/K extends a field of characteristic p , if $\alpha \in K_{\text{sep}}$ is a root of $f_b(x)$, then $f_b(\alpha + n) = 0$ for all $n \in \{1, \dots, p-1\}$. Thus $f_b(x)$ is separable for all $b \in K_{\text{sep}}$ and thus all roots of $f_b(x)$ are contained inside of K_{sep} . It also follows that the map $f_0 : K_{\text{sep}} \rightarrow K_{\text{sep}}$

is a surjective morphism of abelian groups with kernel $\{0, 1, \dots, p-1\} \cong \mathbb{F}_p$, which gives us the exact sequence:

$$0 \rightarrow \mathbb{F}_p \rightarrow K_{\text{sep}} \rightarrow K_{\text{sep}} \rightarrow 0$$

From this we have the long exact sequence induced by Galois cohomology:

$$\dots \rightarrow H^{n-1}(K; K_{\text{sep}}) \rightarrow H^n(K; \mathbb{F}_p) \rightarrow H^n(K; K_{\text{sep}}) \rightarrow \dots$$

By Additive Hilbert 90, $H^n(K; K_{\text{sep}}) = 0$ for all $n \geq 1$, thus the sequence above reduces to $\dots \rightarrow 0 \rightarrow H^n(K; \mathbb{F}_p) \rightarrow 0 \rightarrow \dots$ and we see that $H^n(K; \mathbb{F}_p) = 0$ for all $n > 1$. Note that as abelian groups, $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$, thus proving the result. ■

Corollary 2.1 (N)

For all fields of characteristic p and all finite groups G , $H^n(G; p)_{\text{neg}} = H^n(G; p)$ for all $n > 1$.

Proof:

Let F be a field of characteristic p . By Proposition 1, $H^n(F; p) = 0$ for all $n > 1$, so the induced map $j^* : H^n(G; p) \rightarrow H^n(F; p)$ must be the zero map and thus $H^n(G; p) \subseteq \ker j^*$. The result follows from Proposition 2.3(2) [Gher]. ■

Proposition 2.2

With K as before, $H^n(K; p^\ell) = 0$ for all $n > 1$ and $\ell \in \mathbb{N}$.

Proof:

Proceed by induction using the following sequences:

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$$

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^\ell\mathbb{Z} \rightarrow \mathbb{Z}/p^{\ell-1}\mathbb{Z} \rightarrow 0$$
■

Corollary 2.2 (N)

For fields of characteristic p and all finite groups G , $H^n(G; p^\ell)_{\text{neg}} = H^n(G; p^\ell)$ for all $n > 1$ and $\ell \in \mathbb{N}$.

Proof:

See Corollary 2.1. ■

Proposition 2.3 (N)

For a field of characteristic p and for any p -group G , $H^n(G; \mathbb{Z})_{\text{neg}} = H^n(G; \mathbb{Z})$ for all $n > 2$.

In order to prove this result, we need the following lemma:

Lemma 2.1

The induced multiplication map $m_{|G|} : H^n(G; \mathbb{Z}) \rightarrow H^n(G; \mathbb{Z})$ is the zero map.

Proof:

The trivial group $\{e\} \subseteq G$ satisfies $[G : \{e\}] = |G|$. Using this and restriction-corestriction:

$$\begin{array}{ccccc} H^n(G; \mathbb{Z}) & \xrightarrow{\text{res}} & H^n(\{e\}; \mathbb{Z}) & \xrightarrow{\text{cor}} & H^n(G; \mathbb{Z}) \\ & \searrow & & \nearrow & \\ & & m_{[G:\{e\}]} = m_{|G|} & & \end{array}$$

But $H^n(\{e\}; \mathbb{Z}) = 0$ for all $n \in \mathbb{N}$, so $\text{res} \circ \text{cor} = 0$ by properties of the zero module. ■

Proof of Proposition 2.3:

Consider the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{p^\ell} \mathbb{Z} \rightarrow \mathbb{Z}/p^\ell \mathbb{Z} \rightarrow 0$. With K/F as before, we have that the induced map $m_{p^\ell} := (p^\ell \cdot)_* : H^n(K; \mathbb{Z}) \rightarrow H^n(K; \mathbb{Z})$ is an isomorphism. Using Lemma 2.1 and properties of the inflation map we have the following commuting square for all finite groups G :

$$\begin{array}{ccc} H^n(G; \mathbb{Z}) & \xrightarrow{m_{p^\ell}} & H^n(G; \mathbb{Z}) \\ j_{\mathbb{Z}}^* \downarrow & & \downarrow j_{\mathbb{Z}}^* \\ H^n(K; \mathbb{Z}) & \xrightarrow{\cong} & H^n(K; \mathbb{Z}) \end{array}$$

Take $u \in H^n(G; \mathbb{Z})_{\text{neg}}$. Then $j_{\mathbb{Z}}^*(u) = 0$, so by commutativity of the diagram and the isomorphism on the bottom, $m_{p^\ell}^{-1}(u) \in H^n(G; \mathbb{Z})_{\text{neg}}$. Repeating for each u , we conclude that $m_{p^\ell}^{-1}(H^n(G; \mathbb{Z})) \subseteq H^n(G; \mathbb{Z})_{\text{neg}}$.

Because G is a p -group, $|G| = p^\ell$, thus we have $m_{|G|} = m_{p^\ell} = 0$ from Lemma 2.1, and because $0 \in H^n(G; \mathbb{Z})_{\text{neg}}$, it follows that $m_{p^\ell}^{-1}(0) = H^n(G; \mathbb{Z}) \subseteq H^n(G; \mathbb{Z})_{\text{neg}}$. ■

Although the following results hold in general, we include them here for sake of completion.

Lemma 2.2

Suppose $\gcd(|G|, r) = 1$ for $r \in \mathbb{N}$. Then $H^n(G; r) = 0$.

Proof:

Consider the short exact sequence $0 \rightarrow \mathbb{Z}/r\mathbb{Z} \xrightarrow{|G|} \mathbb{Z}/r|G|\mathbb{Z} \rightarrow \mathbb{Z}/|G|\mathbb{Z} \rightarrow 0$. Using Lemma 1 and the functorial properties of group cohomology, we have the following short exact sequence for all $n \in \mathbb{N}$:

$$0 \rightarrow H^n(G; r|G|) \rightarrow H^n(G; |G|) \rightarrow H^{n+1}(G; r) \rightarrow 0$$

By the Chinese Remainder Theorem, because $\gcd(|G|, r) = 1$ we have that $\mathbb{Z}/r|G|\mathbb{Z} \cong \mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/|G|\mathbb{Z}$, and since group cohomology commutes with direct sums, $H^n(G; r|G|) \cong H^n(G; r) \oplus H^n(G; |G|)$. But since we are working with abelian groups, the only way we can have an injective map $A \oplus B \rightarrow B$ is if $A = 0$, thus $H^n(G; r) = 0$ for all $n \in \mathbb{N}$. ■

Corollary 2.3

Let G be a p -group. Then for any field and any $s \in \mathbb{N}$ such that $p \nmid s$, $H^n(G; s)_{\text{neg}} = H^n(G; s) = 0$ for all $n \in \mathbb{N}$.

Proof:

The result is trivial. ■

With these results we obtain completely compute the negligible group cohomology of p -groups with coefficients in finite abelian groups since direct sums commute with negligible group cohomology. Using the fact that negligible group cohomology commutes with direct limits we are able to compute the negligible group cohomology of degree three and higher when the coefficients are the direct limit of a filtered system.

Consider $\mathbb{Z}/r\mathbb{Z}$ for $r \in \mathbb{N}$. There exist $s, \ell \in \mathbb{N}$ such that $r = sp^\ell$ and $p \nmid s$. By the Chinese Remainder Theorem, $\mathbb{Z}/r\mathbb{Z} \cong \mathbb{Z}/s\mathbb{Z} \oplus \mathbb{Z}/p^\ell\mathbb{Z}$, and because negligible group cohomology preserves direct sums, $H^n(G; r)_{\text{neg}} \cong H^n(G; s)_{\text{neg}} \oplus H^n(G; p^\ell)_{\text{neg}}$.

Define $H_{(p)}^n(G; M) := \{u \in H^n(G; M) : p^\ell \cdot u = 0 \text{ for some } \ell \in \mathbb{N}\}$. We call this the p -torsion subgroup.

Proposition 2.4 (N)

For a field of characteristic p and for any finite group G , $H_{(p)}^n(G; \mathbb{Z}) \subseteq H^n(G; \mathbb{Z})_{\text{neg}}$ for $n > 2$.

Proof:

Because G is finite, $|G| = sp^\ell$, where $s, \ell \in \mathbb{N}$ and $p \nmid s$. Take $P \in \text{Syl}_p(G)$. We have the diagram:

$$\begin{array}{ccccc} H^n(G; \mathbb{Z}) & \xrightarrow{\text{res}} & H^n(P; \mathbb{Z}) & \xrightarrow{\text{cor}} & H^n(G; \mathbb{Z}) \\ & \searrow & & \nearrow & \\ & & m_{[G:P]} = m_s & & \end{array}$$

By Lemma 5.7 [Gher, p. 30], cor preserves negligible classes.. Now, if we restrict res and m_s to $H_{(p)}^n(G; \mathbb{Z})$, then m_s is an isomorphism of $H_{(p)}^n(G; \mathbb{Z})$. Since cor is a homomorphism and everything in $H^n(P; \mathbb{Z})$ is p -torsion for $n > 2$ by Proposition 3, $\text{cor}(H^n(P; \mathbb{Z})) \subseteq H_{(p)}^n(G; \mathbb{Z})$. Thus $\text{cor} \circ \text{res}$ may be thought of as a map $H_{(p)}^n(G; \mathbb{Z}) \rightarrow H_{(p)}^n(G; \mathbb{Z})$ and thus even when restricted in domain and codomain to the p -torsion subgroup, $m_s = \text{cor} \circ \text{res}$. Since $H^n(P; \mathbb{Z})_{\text{neg}} = H^n(P; \mathbb{Z})$ as shown above and since cor preserves negligible classes, $H_{(p)}^n(G; \mathbb{Z}) \subseteq H^n(G; \mathbb{Z})_{\text{neg}}$ for $n > 2$. ■

3. Eventually Negligible Classes

Let R be a unital ring. We will use the shorthand $\mathcal{H}(G; R)$ to denote the anticommutative graded group cohomology ring $(\bigoplus_{n \in \mathbb{N}} H^n(G; R), \smile)$. In a similar manner, $\mathcal{H}(F; R)$ will denote the Galois cohomology ring (which is also anticommutative and graded) for a field F .

Lemma 3.1

Consider the morphisms $f : R \rightarrow S$ and $\phi : G \rightarrow H$, where R, S are unital rings. Then the induced maps $\phi^* : \mathcal{H}(H; R) \rightarrow \mathcal{H}(G; R)$ and $f_* : \mathcal{H}(G; R) \rightarrow \mathcal{H}(G; S)$ are ring homomorphisms.

Proof:

Let $f : R \rightarrow S$ be a ring morphism. We know that $f_* : H^n(G; R) \rightarrow H^n(G; S)$ is a morphism of abelian groups and that in zeroeth degree $f_* \cong f$. Define $f_* : \mathcal{H}(G; R) \rightarrow \mathcal{H}(G; S)$ to act degree-wise. Since f is a ring map and since the identity of $\mathcal{H}(G; R)$ is of degree zero, it follows that f_*

preserves the identity. It is also clearly an abelian group morphism, thus $f_* : \mathcal{H}(G; R) \rightarrow \mathcal{H}(G; S)$ will be a ring homomorphism if the following diagram commutes:

$$\begin{array}{ccc} H^p(G; R) \otimes_{\mathbb{Z}} H^q(G; R) & \xrightarrow{(\cdot) \smile (\cdot)} & H^{p+q}(G \times G; R) \\ f_* \otimes f_* \downarrow & & \downarrow f_* \\ H^p(G; S) \otimes_{\mathbb{Z}} H^q(G; S) & \xrightarrow{(\cdot) \smile (\cdot)} & H^{p+q}(G \times G; S) \end{array}$$

We make use of the cocycle definition of group cohomology. The map $f_* : H^n(G; R) \rightarrow H^n(G; S)$ is induced by $f_{\#} : C^n(G; R) \rightarrow H^n(G; S)$ given by $\varphi \mapsto f \circ \varphi$, where $\varphi \in C^n(G; R)$ is a cocycle. Furthermore, the cup product is induced by the map $\wedge : C^p(G; R) \otimes_{\mathbb{Z}} C^q(G; R) \rightarrow C^{p+q}(G; R)$ given by $(\varphi \wedge \psi)(g_1, \dots, g_{p+q}) = \varphi(g_1, \dots, g_p)\psi(g_{p+1}, \dots, g_{p+q})$, thus it suffices to show that $f_{\#}(\varphi \wedge \psi) = (f_{\#}\varphi) \wedge (f_{\#}\psi)$. Using the fact that f is a ring homomorphism:

$$\begin{aligned} f_{\#}(\varphi \wedge \psi)(g_1, \dots, g_{p+q}) &= f(\varphi(g_1, \dots, g_p)\psi(g_{p+1}, \dots, g_{p+q})) \\ &= f(\varphi(g_1, \dots, g_p))f(\psi(g_{p+1}, \dots, g_{p+q})) \\ &= (f \circ \varphi)(g_1, \dots, g_p)(f \circ \psi)(g_{p+1}, \dots, g_{p+q}) \\ &= (f_{\#}\varphi)(g_1, \dots, g_p)(f_{\#}\psi)(g_{p+1}, \dots, g_{p+q}) \\ &= ((f_{\#}\varphi) \wedge (f_{\#}\psi))(g_1, \dots, g_{p+q}) \end{aligned}$$

Which proves the first part of the result.

Given a group homomorphism $\phi : G \rightarrow H$, we define the induced map $\phi^* : \mathcal{H}(H; R) \rightarrow \mathcal{H}(G; R)$ in a similar manner as f_* . The degree zero part is interpreted to be the degree zero map. By construction this map is an abelian group. Applying the same logic as for f_* and noting that ϕ^* is induced by the map:

$$\begin{aligned} \phi^{\#} : C^n(H; R) &\rightarrow C^n(G; R) \\ \varphi &\mapsto \varphi \circ \left(\underbrace{\phi \times \dots \times \phi}_{n \text{ times}} \right) \end{aligned}$$

The result follows by observing that:

$$\begin{aligned}
& \phi^\#(\varphi \wedge \psi)(h_1, \dots, h_{p+q}) \\
&= (\varphi \wedge \psi) \circ \left(\underbrace{\phi \times \dots \times \phi}_{p+q \text{ times}} \right) (h_1, \dots, h_{p+q}) \\
&= (\varphi \wedge \psi)(\phi(h_1), \dots, \phi(h_{p+q})) \\
&= \varphi(\phi(h_1), \dots, \phi(h_p)) \psi(\phi(h_{p+1}), \dots, \phi(h_{p+q})) \\
&= \left(\varphi \circ \left(\underbrace{\phi \times \dots \times \phi}_p \right) (h_1, \dots, h_p) \right) \left(\psi \circ \left(\underbrace{\phi \times \dots \times \phi}_q \right) (h_{p+1}, \dots, h_{p+q}) \right) \\
&= (\phi^\# \varphi)(h_1, \dots, h_p) (\phi^\# \psi)(h_{p+1}, \dots, h_{p+q}) \\
&= ((\phi^\# \varphi) \wedge (\phi^\# \psi))(h_1, \dots, h_{p+q})
\end{aligned}$$

■

An element $u \in H^n(G; R)$ is said to be eventually negligible iff there exists some $k \in \mathbb{N}$ such that $u^k := \underbrace{u \smile \dots \smile u}_{k \text{ times}} \in H^{nk}(G; R)_{\text{neg}}$.

If R is a ring, $\mathcal{N}(R) := \{a \in R : a^k = 0 \text{ for some } k \in \mathbb{N}\}$.

Corollary 3.1

Given a field F and morphism $j : \Gamma_K \rightarrow G$, $u \in H^n(G; R)$ is eventually negligible iff $j^*(u) \in \mathcal{N}(\mathcal{H}(K; R))$ for all extensions K/F , where $\Gamma_K := \text{Gal}(K_{\text{sep}}/K)$.

Proof:

This follows immediately from Lemma 3.1 and the definition of negligible group cohomology.

■

Lemma 3.2 (N)

Take $n \in \mathbb{N}$. Then for a field F containing n -th roots of unity not of characteristic two and with G a finite group, there exists some $N \in \mathbb{N}$ such that $H^d(G; n)_{\text{neg}} = H^d(G; n)$ for all $d > N$.

Proof:

By the Norm-Residue Isomorphism Theorem, for a field F with n -th roots of unity, $\mathcal{H}(F; n)$ is generated as a $\mathbb{Z}/n\mathbb{Z}$ -algebra by $H^1(F; n)$. But for any $u \in H^1(F; n)$, $u^2 = u \smile u = (-1)^{(1)(1)} u \smile u = -u \smile u = -u^2$, so because $\text{char}(F) \neq 2$ it follows that $u^2 = 0$. Thus $H^1(F; n) \subseteq \mathcal{N}(\mathcal{H}(F; n))$, but because $H^1(F; n)$ generates $\mathcal{H}(F; n)$ as a $\mathbb{Z}/n\mathbb{Z}$ -algebra and since the nilradical is an ideal, $\bigoplus_{m \neq 0} H^m(F; n) \subseteq \mathcal{N}(\mathcal{H}(F; n))$.

Take $u \in H^m(G; n)$ for $m \geq 1$. By Lemma 3.1 and the discussion above, $j^*(u^k) = (j^*(u))^k = 0$ since $j^*(u) \in H^m(F; n)$. It follows from Proposition 2.3(2) [Gher] that $u^k \in H^{mk}(G; n)_{\text{neg}}$, so u is eventually negligible.

We know from Corollary 1.14 [Høg] that $\mathcal{H}(G; n)$ is finitely generated (as a $\mathbb{Z}/n\mathbb{Z}$ -algebra) for each $n \in \mathbb{N}$, so there exists a set $\{u_1, \dots, u_\ell\}$ such that every $u \in \mathcal{H}(G; n)$ may be expressed $u = \sum_{i=1}^m k_i u_1^{j_{1,i}} \dots u_\ell^{j_{\ell,i}}$, where $k_i \in H^0(G; n) \cong \mathbb{Z}/n\mathbb{Z}$. Define $m_i := \min\{r \in \mathbb{N} : u_i^r \text{ is negligible}\}$ and $N := m \cdot \max_i \{m_i\}$. We know m_i is finite for each i since every element of $\mathcal{H}(G; n)$ is negligible in our field F by the discussion above. Take $v \in H^d(G; n)$, where $d > N$. Then $v = \sum_{i=1}^m k_i u_1^{j_{1,i}} \dots u_\ell^{j_{\ell,i}}$, but since the degree of v is greater than N , it follows that there exists at least

one $k' \in \{1, \dots, \ell\}$ such that $j_{k,i} > m_k$. By definition of m_k , it follows that $j^*(u_k^{j_{k,i}}) = 0$. Applying Lemma 3.1, this implies $j^*(v) = \sum_{i=1}^m k_i j^*(u_1^{j_{1,i}}) \dots j^*(u_k^{j_{k,i}}) \dots j^*(u_\ell^{j_{\ell,i}}) = 0$, proving our result. ■

Corollary 3.2

Every element of $H^m(G; n)$ is eventually negligible over fields not of characteristic two containing n -th roots of unity for $m > 1$.

Proof:

By Lemma 3.2, there exists some N such that for $d > N$, every element of $H^d(G; n)$ is negligible. Take $u \in H^m(G; n)$. Then $u^N \in H^{Nm}(G; n)$ must be negligible since $Nm > N$. ■

Proposition 3.1 (N)

Let G be a finite group and F a field containing $|G|$ -th roots of unity. Then there exists some $D \in \mathbb{N}$ such that $H^k(G; \mathbb{Z})_{\text{neg}} = H^k(G; \mathbb{Z})$ for all $k > D$.

Proof:

Consider the sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot|G|} \mathbb{Z} \rightarrow \mathbb{Z}/|G|\mathbb{Z} \rightarrow 0$. Since the induced map $m_{|G|}$ is zero by Lemma 1, the cohomological long exact sequence decomposes into short exact sequences for all $n \geq 1$:

$$0 \rightarrow H^n(G; \mathbb{Z}) \rightarrow H^n(G; |G|) \xrightarrow{\partial_G} H^{n+1}(G; \mathbb{Z}) \rightarrow 0$$

We know from Lemma 3.2 that there exists some $N \in \mathbb{N}$ such that for all $d > N$, $H^d(G; |G|)_{\text{neg}} = H^d(G; |G|)$. By definition of inflation maps and negligible cohomology, ∂_G maps negligible classes into negligible classes. Putting these facts together and using that ∂_G is surjective for sufficiently high degree:

$$\partial_G(H^d(G; |G|)_{\text{neg}}) = \partial_G(H^d(G; |G|)) = H^{d+1}(G; \mathbb{Z}) \subseteq H^{d+1}(G; \mathbb{Z})$$

Setting $D = N + 1$ gives the desired lower bound. ■

Corollary 3.3 (N)

For a field F with $|G|$ -th roots of unity, every element of $H^m(G; \mathbb{Z})$ is eventually negligible for $m > 1$.

Proof:

See Corollary 3.2. ■

4. Connecting Homomorphisms

Given a short exact sequence of abelian groups $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and a group Γ , the connecting homomorphism is the map $H^n(\Gamma; C) \rightarrow H^{n+1}(\Gamma; A)$ in the cohomological exact sequence. In the case of the short exact sequence $0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0$, this map is called the Bockstein homomorphism.

Note that throughout this section we will always assume p to be an odd prime. Let F be a field containing at least the primitive p -th roots of unity. By the Norm-Residue Isomorphism Theorem,

$\mathcal{H}(F; p)$ is generated by elements of $H^1(F; p)$. Let $B : H^n(F; p) \rightarrow H^n(F; p)$ denote the Bockstein homomorphism. It is well-known that $B(\alpha \smile \beta) = B(\alpha) \smile \beta + (-1)^{|\alpha|} \alpha \smile B(\beta)$ for all $\alpha, \beta \in \mathcal{H}(F; p)$. Let $\{\alpha_i\}_{i=1}^\ell$ be a finite set of elements of $H^1(F; p)$. We have the following lemma.

Lemma 4.1 (N)

Let $\{\alpha_i\}_{i=1}^\ell$ be as described above. Then $B(\alpha_1 \smile \dots \smile \alpha_\ell) = -k(\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_\ell$, where (ζ_p) is the class induced by the p -th root of unity.

Proof:

By Lemma 5.2 [Gher], for any $\alpha_1 \in H^1(F; p)$, $B(\alpha_1) = \alpha_1 \smile (\zeta_p) = -(\zeta_p) \smile \alpha_1$.

Now suppose the result holds for $\ell = n$. Then for $\ell = n + 1$ and per properties of Bockstein homomorphism:

$$\begin{aligned}
B(\alpha_1 \smile \dots \smile \alpha_{n+1}) &= B(\alpha_1 \smile \dots \smile \alpha_n) \smile \alpha_{n+1} + (-1)^n \alpha_1 \smile \dots \smile \alpha_n \smile B(\alpha_{n+1}) \\
&= -n(\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} + (-1)^n \alpha_1 \smile \dots \smile \alpha_n \smile \alpha_{n+1} \smile (\zeta_p) \\
&= -n(\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} + (-1)^n (-1)^{n+1} (\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} \\
&= -n(\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} + (-1)^{2n+1} (\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} \\
&= -n(\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} - (\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1} \\
&= -(n+1)(\zeta_p) \smile \alpha_1 \smile \dots \smile \alpha_{n+1}
\end{aligned}$$

Which proves the induction. ■

Corollary 4.1 (N)

The m -th degree Bockstein homomorphism is zero when $p \mid m$.

Proof:

Take $m \in \mathbb{N}$ with $p \mid m$. Then every element of $H^m(F; p)$ is of the form $\sum_{i=1}^m k_i \alpha_{i_1} \smile \dots \smile \alpha_{i_\ell}$ by the Norm Residue Isomorphism Theorem, so by Lemma 4.1 the image under the Bockstein homomorphism is $mA = pA'$ for some $A \in H^{m+1}(F; p)$ and $nA = A'$. But again by the Norm Residue Isomorphism Theorem, everything in $\mathcal{H}(F; p)$ is killed off by p -multiplication, thus $pA' = 0$. ■

Corollary 4.2 (N)

Suppose F contains ζ_{p^2} . Then for all degrees, the Bockstein homomorphism is the zero morphism.

Proof:

By the Norm Residue Isomorphism Theorem, $H^1(F; p) \cong (F^\times)/(F^\times)^p$. Since $\zeta_p = \zeta_{p^2}^p$, it follows that $(\zeta_p) = (\zeta_{p^2}^p) = (\zeta_{p^2})^p = 0$. Since the cup product defines ring multiplication, it follows from the Lemma 4.1 that the Bockstein homomorphism is zero, as the cup product of (ζ_p) with anything must be zero. ■

Proposition 4.1 (N)

Let F be a field and $B : H^{n-1}(K; p) \rightarrow H^n(K; p)$ a Bockstein homomorphism with trivial image for all K/F . Let $i : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z}$ be the canonical “inclusion” map, and let $i_*^G : H^n(G; p) \rightarrow H^n(G; p^2)$ be the induced cohomology morphism for a finite group G . Then over F , $v \in H^n(G; p)_{\text{neg}}$ iff $i_*(v) \in H^n(G; p^2)_{\text{neg}}$.

Proof:

Because $\text{im } B = 0$, the long exact cohomological sequence

$$\dots \rightarrow H^{n-1}(K; p) \xrightarrow{B} H^n(K; p) \rightarrow H^n(K; p^2) \rightarrow \dots$$

Breaks to give a commutative exact square:

$$\begin{array}{ccccc} H^n(G; p) & \xrightarrow{i_*^G} & H^n(G; p^2) & & \\ j^* \downarrow & & \downarrow j^* & & \\ 0 & \longrightarrow & H^n(K; p) & \xrightarrow{i_*^K} & H^n(K; p^2) \end{array}$$

Where j^* is the inflation map induced by a map $j : \Gamma_K \rightarrow G$. Doing a quick diagram chase, suppose $u \in H^n(G; p)_{\text{neg}}$. Then $j^*(u) = 0$, so $i_*^K(j^*(u)) = j^*(i_*^G(u)) = 0$ and $i_*^G(u) \in H^n(G; p^2)_{\text{neg}}$. On the other hand, suppose $i_*^G(u) \in H^n(G; p^2)_{\text{neg}}$. Then $j^*(i_*^G(u)) = i_*^K(j^*(u)) = 0$, but since i_*^K is injective, $j^*(u) = 0$ and thus $u \in H^n(G; p)_{\text{neg}}$. Note that we implicitly used the fact that this holds for all extensions over F . ■

Corollary 4.3 (N)

The image of the Bockstein homomorphism $B^G : H^{n-1}(G; p) \rightarrow H^n(G; p)$ is negligible over a field F provided the Bockstein homomorphism $B : H^{n-1}(K; p) \rightarrow H^n(K; p)$ is trivial for all field extensions K/F .

Proof:

We have the long exact sequence:

$$\dots \rightarrow H^{n-1}(G; p) \xrightarrow{B^G} H^n(G; p) \xrightarrow{i_*^G} H^n(G; p^2) \rightarrow \dots$$

Using Proposition 4.1, $u \in H^n(G; p)_{\text{neg}}$ iff $i_*^G(u) \in H^n(G; p^2)_{\text{neg}}$. But if $u \in \text{im } B^G$, then there exists some $v \in H^{n-1}(G; p)$ such that $B^G(v) = u$, so $i_*^G(u) = i_*^G(B^G(v)) = (i_*^G \circ B^G)(v) = 0$ by properties of chain complexes. But $0 \in H^n(G; p^2)_{\text{neg}}$, thus $u \in H^n(G; p)_{\text{neg}}$ by Proposition 4.1. ■

Corollary 4.4 (N)

If F contains the p^2 -th root unity, then the image of the Bockstein homomorphism is negligible.

Proof:

This follows immediately from Corollaries 4.2 and 4.3 ■

Corollary 4.5 (N)

If $p \mid m$ for $m \in \mathbb{N}$, then the image of the m -th degree Bockstein homomorphism is zero.

Proof:

This follows immediately from Corollaries 4.1 and 4.3. ■

Future Work

Bockstein Homomorphisms

We believe the Bockstein homomorphism may provide additional insights into negligible classes over arbitrary finite groups with coefficients in $\mathbb{Z}/p\mathbb{Z}$. In particular, there are two sets of modules that may prove beneficial to study.

Via well-known properties of the Bockstein homomorphisms, the following is a chain complex for any group G :

$$\dots \rightarrow H^{n-1}(\Gamma; p) \xrightarrow{B^{n-1}} H^n(\Gamma; p) \xrightarrow{B^n} H^{n+1}(\Gamma; p) \rightarrow \dots$$

Thus we have the cohomology modules $H^n(\Gamma)_p := \ker B^n / \operatorname{im} B^{n-1}$. These are first set of potentially interesting modules.

Let F be a field with absolute Galois group Γ_F and G a finite field. Then given $j : \Gamma_F \rightarrow G$, we have an induced map $\tilde{j} : H^n(G)_p \rightarrow H^n(\Gamma_F)_p$, which may provide insight into the map $j^* : H^n(G; p) \rightarrow H^n(F; p)$ and thus into the negligible classes of G over F .

With B again denoting the Bockstein homomorphism, the second set of potentially interesting modules are $\mathcal{B}^n(G, F)_p := H^n(G; p)_{\operatorname{neg}, F} / (\operatorname{im} B^{n-1} \cap H^n(G; p)_{\operatorname{neg}, F})$. In particular, under the assumptions of Corollaries 4.4 or 4.5, $\operatorname{im} B^{n-1} \cap H^n(G; p)_{\operatorname{neg}, F} = \operatorname{im} B^{n-1}$, in which case $\mathcal{B}^n(G; F)_p$ measures the failure of the Bockstein homomorphism to account for all negligible classes. By exploring properties of these modules (specifically homological ones), we hope to be able to quantify in a meaningful way the failure of the Bockstein homomorphism to provide a natural description of negligible group cohomology.

Generalized Negligible Classes

Let G, H be topological groups and M an abelian group. Define the modules:

$$\operatorname{Neg}_M^n(G, H) := \bigcap_{f \in \operatorname{Hom}_{\operatorname{Top}}(G, H) \cap \operatorname{Hom}_{\operatorname{Grp}}(G, H)} \ker(f^* : H^n(H; M) \rightarrow H^n(G; M))$$

With this definition, note that for finite groups G :

$$H^n(G; M)_{\operatorname{neg}, F} = \bigcap_{K/F} \operatorname{Neg}_M^n(\Gamma_K, G)$$

Thus, any results concerning $\operatorname{Neg}_M^n(G, H)$ may be almost immediately applied to the study of negligible classes.

This construction also allows for a notion of “profinite negligible cohomology,” although this may require using Galois cohomology in place of group cohomology. Computations of profinite negligible classes could provide examples of groups that fail to have a solution to the Inverse Galois Problem.

Finally, if G is a group with normal subgroup N , studying $\text{Neg}_M^n(G, G/N)$ could provide other relations between $H^n(G; M)_{\text{neg}}$ and $H^n(G/N; M)_{\text{neg}}$ since for a field F with absolute Galois group Γ_F , the maps $G \rightarrow G/N$ induce maps $\Gamma \rightarrow G/N$.

Other Computations of Negligible Classes

In Theorem 5.3 [Gher], the negligible classes of $\mathcal{H}((\mathbb{Z}/p\mathbb{Z})^r; p)$ are completely computed. We would like to extend these computations to $\mathbb{Z}/p^2\mathbb{Z}$ coefficients.

We also wish to extend the computation of $H^2(p^k; p^\infty)_{\text{neg}}$ to fields that properly contain all p^k -th roots of unity.

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