

Basics of (Pre)Sheaves

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Resources

Ulrich Görtz and Torsten Wedhorn, *Algebraic Geometry 1: Schemes*, Chapter 2

John M. Lee, *Introduction to Complex Manifolds*, Chapter 5

Section 1: Presheaves and Sheaves

Given a topological space X , let $\mathcal{U}(X)$ denote the poset category of open sets of X . All categories will be assumed to have a zero object and all small products.

Definition 1.1 (Presheaf)

A **presheaf** on X over a category $\mathcal{C}$ is a functor $\mathcal{F} : \mathcal{U}(X)^{op} \rightarrow \mathcal{C}$ s.t. $\mathcal{F}(\emptyset) = 0$. Morphisms are natural transformations. We will denote this by $\text{PSh}(X)_{\mathcal{C}}$, or $\text{PSh}(X)$ if $\mathcal{C}$ is clear.

Example 1.1

Given $A \in \text{Ob}(\mathcal{C})$, the constant presheaf $\underline{A} \in \text{PSh}(X)_{\mathcal{C}}$ is defined to be $\mathcal{A}(U) = \begin{cases} A & \text{if } U \neq \emptyset \\ 0 & \text{if } U = \emptyset \end{cases}$.

Proposition 1.1

If $\mathcal{C}$ is (co)complete, then so is $\text{PSh}(X)_{\mathcal{C}}$.

Proof:

Define pointwise, i.e., $(\lim_{i \in I} F_i)(U) := \lim_{i \in I} F_i(U)$ and $(\text{colim}_{i \in I} F_i)(U) = \text{colim}_{i \in I} F_i(U)$.

■

Definition 1.2 (Sheaf)

A **sheaf** $\mathcal{F}$ on X over $\mathcal{C}$ is a presheaf over $\mathcal{C}$ satisfying the **gluing/sheaf condition**: for any open cover $\{U_i : i \in I\}$ of $U \subseteq X$ open, the following is an equalizer

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$$

The category of sheaves is the full subcategory of $\text{PSh}(X)_{\mathcal{C}}$ spanned by presheaves satisfying the gluing condition and is denoted $\text{Sh}(X)_{\mathcal{C}}$, or just $\text{Sh}(X)$ if $\mathcal{C}$ is clear. A **section** of $\mathcal{F}$ over U is an element $s \in \mathcal{F}(U)$. If $V \subseteq U$, then $s|_V$ is the image of s under the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$.

Example 1.2

The constant presheaf $\underline{\mathbb{Z}}$ in $\text{Sh}(\mathbb{R})_{\mathbb{Z}}$ is not a sheaf. Consider the open set $U = (1, 2) \cup (2, 3)$. Then $\{(1, 2), (2, 3)\}$ is an open cover of U . Take $1 \in \mathcal{A}((1, 2))$, $2 \in \mathcal{A}((2, 3))$. These maps both restrict to 0 on $(1, 2) \cap (2, 3) = \emptyset$, so they agree on overlaps. But no section of $\mathcal{A}(U) = \mathbb{Z}$ restricts to 1 on $(1, 2)$ and 2 on $(2, 3)$, since the other restriction maps are just identities.

Let X be a complex manifold. The sheaf $\mathcal{O}_X(U) := \{\text{holomorphic functions } U \rightarrow \mathbb{C}\}$ is a sheaf.

The sheaf of regular functions over a variety.

Definition 1.3 (Stalk)

Let $\mathcal{C}$ be closed under filtered colimits. Given $x \in X$ and $\mathcal{F} \in \mathbf{Ob}(\mathbf{PSh}(X)_{\mathcal{C}})$, we define the stalk of $\mathcal{F}$ at x to be

$$\mathcal{F}_x := \operatorname{colim}_{U \ni x} \mathcal{F}(U)$$

This defines a functor $\mathbf{PSh}(X)_{\mathcal{C}} \rightarrow \mathcal{C}$. Explicitly,

$$\mathcal{F}_x = \{(s, U) : U \ni x \text{ open}, s \in \mathcal{F}(U)\} / \sim$$

Where $\sim$ is the equivalence relation where $(s_1, U_1) \sim (s_2, U_2)$ if there exists $W \subseteq U_1 \cap U_2$ open such that $s_1|_W = s_2|_W$. For $U \ni x$ is a map $\mathcal{F}(U) \rightarrow \mathcal{F}_x$ sending $s \mapsto s_x := \langle s, U \rangle$. We call this the **space of germs**.

If $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism, then under this identification with the space of germs the functoriality manifests as $\phi_x(\langle s, U \rangle) = \langle \phi_U(s), U \rangle$.

Example 1.3

The stalk of the constant presheaf $\underline{\mathbb{Z}}$ is just $\mathbb{Z}$ (the colimit of a constant functor is the evaluation of the functor).

Let $\mathcal{C}$ be $R\text{-mod}$, or $R\text{-alg}$ for R a commutative ring.

Proposition 1.2

A morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ in $\mathbf{Sh}(X)$ is injective (isomorphism) on all open sets U iff φ_x is injective (resp. isomorphism) for all $x \in X$.

Proof:

Consider the following map of sheaves

$$\begin{aligned} \Phi : \mathcal{F}(U) &\rightarrow \prod_{x \in U} \mathcal{F}_x \\ s &\mapsto (s_x)_{x \in U} \end{aligned}$$

Let $s, t \in \mathcal{F}(U)$ be such that $s_x = t_x$ for all $x \in U$. Then for each $x \in U$, there exists a nbhd W_x such that $s|_{W_x} = t|_{W_x}$. By the sheaf condition, these must be equal, so Φ is injective. Now consider the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \hookrightarrow & \prod_{x \in U} \mathcal{F}_x \\ \varphi_U \downarrow & & \downarrow \prod_x \varphi_x \\ \mathcal{G}(U) & \hookrightarrow & \prod_{x \in U} \mathcal{G}_x \end{array}$$

Then if $\prod_x \varphi_x$ is injective, so is $\hookrightarrow \circ \varphi_U$, this φ_U is injective. For the other direction, we have that the filtered colimit and product of injective maps is injective.

If φ_U is an isomorphism for all $U \subseteq X$, by functoriality of products and filtered colimits we have that $\prod_x \varphi_x$ is an isomorphism. Thus it suffices to show that if φ_x is an isomorphism for all $x \in X$, then φ_U is surjective for all $U \subseteq X$ open. Pick $t \in \mathcal{G}(U)$. For $x \in U$, pick $U^x \ni x$ and $s^x \in \mathcal{F}(U^x)$ such that $\varphi_x(s^x) = (\varphi_{U^x}(s^x))_x = t_x$. Then there exists open $V^x \subseteq U^x$ such that $\varphi_{V^x}(s^x|_{V^x}) = t|_{V^x}$. Then V^x is an open cover of U and

$$\varphi_{V^x \cap V^y}(s^x|_{V^x \cap V^y}) = t|_{V^x \cap V^y} = \varphi_{V^x \cap V^y}(s^y|_{V^x \cap V^y})$$

Since $\varphi_{V^x \cap V^y}$ is injective, $s^x|_{V^x \cap V^y} = s^y|_{V^x \cap V^y}$, so by the sheaf condition, the $s^x|_{V^x}$ glue together into $s \in \mathcal{F}(U)$. Since $s|_{V_x} = s^x|_{V_x} = t|_{V_x}$, again by the gluing condition $\varphi(s) = t$. ■

Definition 1.4 (Injective, Surjective)

A morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is sheaves over X is **injective** (**surjective**) if φ_x is injective (resp. surjective) for all $x \in X$.

Caution!

Even if φ is surjective on all stalks, it may not be surjective on open sets. This is, in some sense, the idea driving sheaf cohomology.

Section 2: Some Category Theory

Definition 2.1 (Yoneda embedding)

Given a category $\mathcal{C}$, there is a fully faithful embedding $\mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Set})$ sending $c \mapsto \text{Hom}_{\mathcal{C}}(-, c)$. The category $\text{Fun}(\mathcal{C}^{op}, \text{Set})$ is often called the category of **presheaves over $\mathcal{C}$** and is denoted $\hat{\mathcal{C}}$.

Definition 2.2 (Adjunction)

Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$ functors. We say that F is **left adjoint** to G (resp. G is **right adjoint** to F), denoted $F \dashv G$, if there exists a natural transformation

$$\text{Hom}_{\mathcal{D}}(F(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(G(-), -)$$

Equivalently, $F \dashv G$ if there exist natural transformations $\eta : \text{Id}_{\mathcal{C}} \Rightarrow GF, \varepsilon : FG \Rightarrow \text{Id}_{\mathcal{D}}$ (called the **unit** and **counit**, resp.) such that the following compositions are the corresponding identity natural transformations

$$F \xRightarrow{F\eta} FGF \xRightarrow{\varepsilon F} F$$

$$G \xRightarrow{\eta G} GFG \xRightarrow{G\varepsilon} G$$

Given definition 1, define $\eta : \text{Id}_{\mathcal{C}} \Rightarrow GF, \eta : FG \Rightarrow \text{Id}_{\mathcal{D}}$ by having η_X be the morphism corresponding to $\text{id}_{F(X)}$ and ε_Y that corresponding to $\text{id}_{G(Y)}$ for any $X \in \text{Ob}(\mathcal{C}), Y \in \text{Ob}(\mathcal{D})$.

Given definition 2, define $\Phi : \text{Hom}_{\mathcal{D}}(F(-), -) \rightarrow \text{Hom}_{\mathcal{C}}(-, G(-))$ by

$$\text{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{G} \text{Hom}_{\mathcal{C}}(GF(X), G(Y)) \xrightarrow{\eta_X^*} \text{Hom}_{\mathcal{C}}(X, G(Y))$$

This gives the equivalence of definitions.

Example 2.1

In $R\text{-mod}$, $\text{Hom}_R(M \otimes_R N, L) \cong \text{Hom}_R(N, \text{Hom}_R(M, L))$.

Any colimit in $\mathcal{C}$ over a small category I is a left adjoint to the constant functor, that is

$$\text{Hom}_{\mathcal{C}}(\text{colim}_{i \in I} F, d) \cong \text{Hom}_{\mathcal{C}^I}(F, \text{const}(d))$$

Similarly, any limit in $\mathcal{C}$ over a small category I is right adjoint to the constant functor.

Some Facts

We have $\text{Hom}(\text{colim}(-), -) \cong \lim \text{Hom}(-, -)$ and $\text{Hom}(-, \lim(-)) \cong \lim \text{Hom}(-, -)$.

Left (resp. right) adjoints are unique up to unique isomorphism.

Proposition 2.1

Left adjoints preserve colimits, and right adjoints preserve limits.

Proof:

Let $F \dashv G$, where $F : \mathcal{C} \rightarrow \mathcal{D}$. Let $d \in \mathcal{D}$ be arbitrary. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(F(\mathrm{colim}_{i \in I} c_i), d) &\cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{i \in I} c_i, G(d)) \\ &\cong \lim_{i \in I} \mathrm{Hom}_{\mathcal{C}}(c_i, G(d)) \\ &\cong \lim_{i \in I} \mathrm{Hom}_{\mathcal{D}}(F(c_i), d) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(\mathrm{colim}_{i \in I} F(c_i), d) \end{aligned}$$

Each isomorphism is natural. Now use the Yoneda embedding (Definition 2.1). ■

Remark 2.1

A more careful proof of Proposition 2.1 can be done to show that if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a left-adjoint, then $(F(\mathrm{colim}_{i \in I} X_i), \iota_i)$ is the universal cocone of the system $(F(X_i), i \in I)$. In particular, this gives a way to prove existence of colimits.

Corollary 2.1

Colimits commute with colimits, and limits commute with limits.

Proof:

Colimits are left adjoints, now apply Proposition 2.1. ■

Section 3: Sheafification

Proposition 3.1

If $\mathcal{C}$ is complete, then $\mathrm{Sh}(X)_{\mathcal{C}}$ is complete.

Proof:

From Proposition 1.1, we know that limits over presheaves exist and are given pointwise. Taking the limit of the corresponding equalizer diagrams defining the gluing conditions are still equalizers by Corollary 2.1, thus the pointwise limit is a sheaf. ■

The same argument does not work for presheaves. To resolve this, we want a functor which turns a presheaf into a sheaf. Intuitively, the idea of the construction is to replace “global” properties by “local” properties. As we will see below, this corresponds to “approximating” a presheaf by sheaves.

Definition 3.1 (Sheafification)

*The **sheafification functor** is the functor $\mathrm{sh} : \mathrm{PSh}(X) \rightarrow \mathrm{Sh}(X)$ which is left adjoint to the inclusion functor $i : \mathrm{Sh}(X) \rightarrow \mathrm{PSh}(X)$.*

Proposition 3.2

The sheafification functor exists, the unit is an isomorphism on stalks, and the sheafification of a sheaf is naturally isomorphic to itself.

Proof:

Defining the sheafification functor

Given $U \subseteq X$ open and a presheaf $\mathcal{F} \in \text{Sh}(X)$, define

$$\text{sh}(\mathcal{F})(U) = \tilde{\mathcal{F}}(U) := \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x : \forall x \in U, \exists \text{ open nbhd } x \in W \subseteq U \right. \\ \left. \text{and } t \in \mathcal{F}(W) \text{ s.t. } \forall w \in W, s_w = \langle t, W \rangle_w \right\}$$

For $V \subseteq U$, the restriction maps are the induced by the projection $\prod_{x \in U} \mathcal{F}_x \rightarrow \prod_{x \in V} \mathcal{F}_x$. The sheaf condition follows easily from writing everything out (intuitively, it holds since the information is purely local). Define the unit map $\eta : \text{Id}_{\text{PSh}(X)} \Rightarrow i \circ \text{sh}$

$$\eta_{\mathcal{F}, U} : \mathcal{F}(U) \rightarrow \tilde{\mathcal{F}}(U)$$

$$s \mapsto (\langle s, U \rangle_x)_{x \in U}$$

Where $\langle s, U \rangle_x$ denotes the germ of s at $x \in U$.

Let $\phi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ be a morphism of presheaves. Define $\text{sh}(\phi) : \text{sh}(\mathcal{F}_1) \rightarrow \text{sh}(\mathcal{F}_2)$ by

$$\text{sh}(\phi)_U : \text{sh}(\mathcal{F}_1)(U) \rightarrow \text{sh}(\mathcal{F}_2)(U)$$

$$(s_x)_{x \in U} \mapsto (\phi_x(s_x))_{x \in U}$$

It can be checked that $\eta_{\mathcal{F}}$ is a morphism of presheaves, that η defines a natural transformation, and that $\text{sh}(\phi)$ preserves compositions and identities and thus is a functor.

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Unit is isomorphism on stalks

Fix $z \in X$. First note that

$$\tilde{\mathcal{F}}_z = \left\{ (\langle s_x \rangle_{x \in U}, U) : U \ni z \text{ open}, (s_x)_{x \in U} \in \tilde{\mathcal{F}}(U) \right\}$$

Take some $\langle s_x \rangle_{x \in U}, U$. By the definition of $\tilde{\mathcal{F}}$, there exists some $U \supseteq W \ni z$ and $t \in \mathcal{F}(W)$ such that for all $w \in W$, $s_w = \langle t, W \rangle_w$. In particular, this means

$$\langle s_x \rangle_{x \in U}, U = \langle (\langle t, W \rangle_w)_{w \in W}, W \rangle$$

For $\langle s, U \rangle_z \in \mathcal{F}_z$, we have

$$\eta_{\mathcal{F}, z}(\langle s, U \rangle_z) = \langle \eta_{\mathcal{F}, U}(s), U \rangle_z = \langle (\langle s, U \rangle_x)_{x \in U}, U \rangle_z$$

Thus $\eta_{\mathcal{F}, z}$ is surjective. On the other hand, if

$$\langle (\langle s, U \rangle_x)_{x \in U}, U \rangle_z = \langle (\langle t, V \rangle_y)_{y \in V}, V \rangle_z$$

Then there exists $W \subseteq U \cap V$ such that $\langle s, U \rangle_y = \langle t, V \rangle_y$ for all $y \in U \cap V$. In particular, $z \in U \cap V$, thus $\langle s, U \rangle_z = \langle t, V \rangle_z$, thus $\eta_{\mathcal{F},z}$ is surjective, so $\eta_{\mathcal{F}}$ induces an isomorphism on all stalks.

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Sheafification preserves sheaves

If $\mathcal{G}$ is a sheaf, then since $\eta_{\mathcal{G},x}$ is an isomorphism for each $x \in X$, by Proposition 1.2 we have that $\eta_{\mathcal{G}} : \mathcal{G} \rightarrow \tilde{\mathcal{G}}$ is an isomorphism in $\text{Sh}(X)$. With this in mind, define the counit map $\varepsilon : \text{sh} \circ i \Rightarrow \text{Id}_{\text{Sh}(X)}$ by $\varepsilon_{\mathcal{G}} := \eta_{\mathcal{G}}^{-1}$, viewed as a map in $\text{Sh}(X)$.

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Sheafification is the left adjoint to inclusion

It remains to check that the adjunction property is satisfied.

Note that if $\mathcal{G}$ is a sheaf, then $\eta_{\mathcal{G}}, \eta_{i(\mathcal{G})}, i(\eta_{\mathcal{G}})$ all make the same assignments (as do their inverses) and thus may be treated (in an abuse of notation) as the same map.

Now we check that the adjunction diagrams are satisfied. Let $\mathcal{G}$ be a sheaf. Then

$$i(\varepsilon_{\mathcal{G}}) \circ \eta_{i(\mathcal{G})} = i(\eta_{\mathcal{G}}^{-1}) \circ \eta_{i(\mathcal{G})} = i(\eta_{\mathcal{G}}^{-1}) \circ i(\eta_{\mathcal{G}}) = i(\eta_{\mathcal{G}}^{-1} \circ \eta_{\mathcal{G}}) = i(\text{id}_{\mathcal{G}}) = \text{id}_{i(\mathcal{G})}$$

And for any presheaf $\mathcal{F}$

$$\varepsilon_{\text{sh}(\mathcal{F})} \circ \text{sh}(\eta_{\mathcal{F}}) = \eta_{\text{sh}(\mathcal{F})}^{-1} \circ \text{sh}(\eta_{\mathcal{F}})$$

It suffices to show that $\text{sh}(\eta_{\mathcal{F}})$ makes the same identifications as $\eta_{\text{sh}(\mathcal{F})}$, as we can then view these as the same map and we get that the composition is $\text{id}_{\text{sh}(\mathcal{F})}$. This is an exercise in unpacking definitions.

■

Exercise:

Finish the proof of Proposition 3.2 by showing that $\eta_{\mathcal{F}}$ is a map of presheaves, that η defines a natural transformation, and that sh preserves composition and identities. Unpack definitions to show that $\text{sh}(\eta_{\mathcal{F}})$ and $\eta_{\text{sh}(\mathcal{F})}$ make the same identifications.

Using the equivalence of definitions of adjunction, the isomorphism

$$\text{Hom}_{\text{Sh}(X)}(\tilde{\mathcal{F}}, \mathcal{G}) \rightarrow \text{Hom}_{\text{PSh}(X)}(\mathcal{F}, i(\mathcal{G}))$$

Is given by

$$\varphi \mapsto i(\varphi) \mapsto i(\varphi) \circ \eta_{\mathcal{F}} = \varphi \circ \eta_{\mathcal{F}}$$

I.e., any map $\tilde{\varphi} : \mathcal{F} \rightarrow i(\mathcal{G})$ can be uniquely factored as $\tilde{\varphi} = \varphi \circ \eta_{\mathcal{F}}$ for some $\varphi : \mathcal{F} \rightarrow \mathcal{G}$.

Corollary 3.1

If $\mathcal{C}$ is cocomplete, then so is $\text{Sh}(X)_{\mathcal{C}}$.

Proof:

Let $I \rightarrow \text{Sh}(X)_{\mathcal{C}}$ be a system of sheaves. By Remark 2.1, $\text{sh}(\text{colim}_{i \in I} X_i)$ satisfies the universal property of the system $\text{sh}(X_i) \cong X_i$.

■

Now that these tools are in place, here is another perspective on sheafification. By the theory of Kan extensions

$$\mathrm{sh}(\mathcal{F}) = \mathrm{Lan}_i(\mathrm{Id}_{\mathrm{Sh}(X)})(\mathcal{F}) = \mathrm{colim}_{x \in i/\mathcal{F}}(\mathrm{Id}_{\mathrm{Sh}(X)})$$

Where $i/\mathcal{F}$ is the comma category whose objects are pairs $(\mathcal{G}, i(\mathcal{G}) \rightarrow \mathcal{F})$ with $\mathcal{G}$ is a sheaf, and whose morphisms are those $\phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ for which $(i(\mathcal{G}_2) \rightarrow \mathcal{F}) \circ \phi = (i(\mathcal{G}_1) \rightarrow \mathcal{F})$. So intuitively, the sheafification of a presheaf $\mathcal{F}$ is the sheaf obtained from “gluing together” all “approximations” of $\mathcal{F}$ by sheaves. We can also recover the properties of Proposition 3.2. If $\mathcal{F} \in \mathrm{Ob}(\mathrm{Sh}(X)_C)$, then $i/\mathcal{F}$ has terminal object $\mathcal{F} \xrightarrow{\mathrm{id}} \mathcal{F}$, thus $\mathrm{colim}_{x \in i/\mathcal{F}}(\mathrm{Id}_{\mathrm{Sh}(X)})$ is the evaluation of $\mathrm{Id}_{\mathrm{Sh}(X)}$ at the terminal object, which is equal to $\mathcal{F}$. Using the Yoneda lemma and abstract nonsense, it can also be shown that the unit of adjunction induces an isomorphism on stalks.

Here is yet another perspective on sheafification: it is the universal functor which makes two things which “look the same” locally also “look the same” globally. Define a *local equivalence* of presheaves to be a map $\phi : \mathcal{F} \rightarrow \mathcal{G}$ such that $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is an isomorphism for all $x \in X$. If $\mathcal{F}, \mathcal{G}$ are not sheaves, this will not hold true in general (recall how the sheaf condition was used in Proposition 1.2). Denote this class of maps by S . By Proposition 1.2 and Proposition 3.2, sheafification maps local equivalences to isomorphisms. Furthermore, it is universal, in the sense that precomposition with sheafification yields for any category D an equivalence

$$\mathrm{Fun}(\mathrm{Sh}(X), D) \rightarrow \mathrm{Fun}^S(\mathrm{PSh}(X), D)$$

Where $\mathrm{Fun}^S(\mathrm{PSh}(X), -)$ is the category of functors which map elements of S to isomorphisms in D . We call $\mathrm{sh} : \mathrm{PSh}(X) \rightarrow \mathrm{Sh}(X)$ the *localization* of $\mathrm{PSh}(X)$ by S .

Example 3.1

Using the construction in Proposition 3.2, the sheafification of the constant functor $\underline{\mathbb{Z}} \in \mathrm{PSh}(X)$ is given by

$$\mathrm{sh}(\underline{\mathbb{Z}})(U) = \left\{ (z_x) \in \prod_{x \in U} \mathbb{Z} : \forall x \in U, \exists \text{ open nbhd } x \in W \subseteq U \text{ and } t \in \mathcal{F}(W) \text{ s.t. } \forall w \in W, z_w = t \right\}$$

Which we see can be identified the the space of locally constant functions on U .

Section 4: Direct and Inverse Image Sheaves

Definition 4.1 (Direct image functor)

Let $f : X \rightarrow Y$ be continuous. The **direct image functor** $f_* : \mathrm{PSh}(X) \rightarrow \mathrm{PSh}(Y)$ is defined to be

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

Note that if $g : Y \rightarrow Z$ is another continuous function, $(g \circ f)_* = g_* \circ f_*$

Proposition 4.1

The direct image functor restricts to a functor on sheaves.

Proof:

We just need to check the gluing condition. But this follows easily, since for any open $U \subseteq Y$ and open cover $\{U_i : i \in I\}$ of U , $\{f^{-1}(U_i) : i \in I\}$ is an open cover of $f^{-1}(U)$, and $\mathcal{F}$ satisfies the sheaf condition w.r.t. this cover. ■

Definition 4.2 (Inverse image functor)

Let $f : X \rightarrow Y$ be a continuous maps. The **inverse image functor** $f^{-1} : \text{Sh}(Y) \rightarrow \text{Sh}(X)$ is defined as follows. For $\mathcal{G} \in \text{Sh}(Y)$, define a presheaf

$$f^+\mathcal{G}(U) := \text{colim}_{\substack{V \supseteq f(U) \\ V \text{ open}}} \mathcal{G}(V)$$

then $f^{-1}\mathcal{G} := \text{sh}(f^+\mathcal{G})$

Note that $(f^{-1}\mathcal{G})_x \cong (f^+\mathcal{G}(U))_x \cong \mathcal{G}(U)_{f(x)}$. The direct and inverse image functors form an adjunction pair

$$\text{Hom}_{\text{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\text{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$$