

Sheaf-Theoretic Framework for Ruled Coloring Problems

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A coloring of a graph is a function from the vertices of a graph to the set of natural numbers. A “ruled coloring” is a collection of colorings satisfying a certain property, called the “rule.” For example, a rule could be that any two adjacent vertices are assigned different values. In this paper, we formalize these notions and construct a framework allowing logical operations to be performed on rules. We begin by discussing the category of graphs with embeddings. We introduce a family of topologies on the category of graphs with embeddings and show that the set of colorings defines a sheaf for any of these topologies. From here we define rules and ruled colorings and discuss the interpretation of Joyal-Kripke semantics for rules. Finally, we show how proper colorings and abelian group-valued colorings fit into this framework, and we suggest how cohomological tools could be used to solve such problems.

1 Introduction and Motivation

Sheaf theory is a powerful tool in mathematics generalizing the idea that “global information” can be deduced by “local information.” Given a category with a notion of “topology” on each of its objects, a sheaf is a presheaf (i.e. a contravariant functor from that category to the category of sets) which can be computed by gluing together the values of the functor on “open pieces.” A precise definition is given in the appendix (Section 5). The category of these sheaves is a Grothendieck topos. In addition to the aforementioned geometric properties, these have a logical aspect: one of the objects of the topos (called the subobject classifier) admits morphisms which give it the structure of a Heyting algebra (i.e. a model for intuitionistic propositional logic). The “internal” logic (i.e. the structure of the subobject classifier) can be used to deduce “external” statements (i.e. statements about the way the objects of the topos behave). These ideas are explored in depth in [3, Chapters IV.8, VI.7], and the interplay between the internal and external viewpoints has been exploited in various areas of mathematics. Some interesting examples of this in algebraic geometry are found in [1] and are the main source of motivation for this paper. Crudely speaking, we aim to make precise the notion of a “ruled coloring” of a graph using sheaf theory. Once this is established, we explore some basic properties of this framework, suggest ways in which it could be applied to problems in graph theory, and discuss what the internal language of the topos implies in this setting. Due to the somewhat concrete nature of graph theory, we hope that this provides an accessible introduction and intuition to some of the basic ideas behind sheaf and topos theory.

Intuitively speaking, a coloring of a graph is simply an assignment of “colors” (i.e. elements of a set) to the vertices of a graph. A ruled coloring ought to be some sort of “restriction” on ways in which the colors can be assigned. This restriction should be something applicable to every (or at the very least some distinguished family) graph. These could be of two natures: local or global; the former consists of rules about neighbors of a vertex, and the latter rules about the whole graph. For example, we could have the rule “every vertex of the graph has a red neighbor” (a local rule) or “the graph has at least one red vertex” (a global rule). Altogether, this suggests that we should view ruled colorings as subsheaves of a sheaf of all possible colorings. The local versus global rules will depend on the choice of topology we place on the graphs. The logical structure of a topos mentioned above allows us to generate a propositional logic on ruled colorings. In this way, we can describe in a precise sense how to conjoin rules (i.e. rule A and rule B, rule A or rule B, etc.). This is explored further in Section 3.

We begin this paper by giving our definition of the category of graphs. Following this, we discuss a family of topologies that can be placed on this category. In the next section, we introduce the formal definition of ruled colorings. We discuss the propositional logic on ruled colorings from the “internal” and “external” perspectives. As the internal logic of many topoi is intuitionistic, we spend some time discussing what the negation of a ruled coloring is. We then have a brief conversation on parameterized ruled colorings, i.e. rules that vary depending on what the set of colors is. In the next section, we discuss a few combinatorial problems and reformulate them in this sheaf-theoretic framework. Finally, we discuss the machinery of sheaf cohomology as a means to count the number of a specific ruled colorings of a graph, and we propose a combinatorial problem for which these techniques are likely to be insightful. A brief overview of the relevant tools from topos theory and homological algebra is included in the appendix.

We will assume the reader is familiar with the machinery and notation of basic category theory and graph theory. For notation, let C_n denote the n -cycle graph for $n \geq 3$. The category of sets is denoted $\mathbf{Set}$, abelian groups by $\mathbf{Ab}$, and finite abelian groups by $\mathbf{Ab}^{\text{fin}}$.

2 Topologies on Categories of Graphs

This section sets up the overarching categorical framework required to discuss the sheaf-theoretic approach to colorings of Section 3. In 2.1 we define the category of interest for this paper (undirected graphs with embeddings) and show that it is closed under pullbacks. A collection of Grothendieck topologies are defined in 2.2; these will be the topologies on which the set of colorings of a graph forms a sheaf.

2.1 The Category Graph

An undirected graph G consists of a set $V(G)$ (the vertices) and a set $E(G)$ of unordered pairs of elements in $V(G)$ (the edges) such that for each $g \in V(G)$, $(g, g) \in E(G)$. A *morphism* $\phi : G \rightarrow H$ of graphs is a pair of functions $\phi_V : V(G) \rightarrow V(H)$, $\phi_E : E(G) \rightarrow E(H)$ such that for any $(g_1, g_2) \in E(G)$, $\phi_E(g_1, g_2) = (\phi_V(g_1), \phi_V(g_2)) \in E(H)$. Note that ϕ_E is uniquely determined by ϕ_V . We also require that ϕ_V is injective. This condition forces ϕ_E

to be injective, for if $\phi_E(g_1, g_2) = (\phi_V(g_1), \phi_V(g_2)) = (\phi_V(g'_1), \phi_V(g'_2)) = \phi_E(g'_1, g'_2)$, then $\phi_V(g_1) = \phi_V(g'_1)$ and $\phi_V(g_2) = \phi_V(g'_2)$, which by injectivity implies $g_1 = g'_1, g_2 = g'_2$, thus $(g_1, g_2) = (g'_1, g'_2)$. We will sometimes call these maps embeddings. Morphisms of graphs can be composed in the obvious way, thus there is a category **Graph** whose objects are graphs and whose morphisms are morphisms of graphs. Let **Graph** denote the corresponding category and **Graph**^{fin} the full subcategory spanned by finite graphs (i.e. graphs G with $|V(G)| < \infty$).

Given graphs G_1, G_2, G and morphisms $\phi_1 : G_1 \rightarrow G, \phi_2 : G_2 \rightarrow G$, the pullback graph $G_1 \times_G G_2$ is defined as

$$V(G_1 \times_G G_2) := V(G_1) \times_{V(G)} V(G_2) = \{(g_1, g_2) \in V(G_1) \times V(G_2) \mid \phi_1(g_1) = \phi_2(g_2)\}$$

$$E(G_1 \times_G G_2) := \{((g_1, g_2), (g'_1, g'_2)) \in E(G_1) \times E(G_2) \mid (g_1, g'_1), (g_2, g'_2) \in V(G_1 \times_G G_2)\}$$

Note that for $i = 1, 2$ there are morphisms $\pi_i : G_1 \times_G G_2 \rightarrow G_i$ which project the elements of $V(G_1 \times_G G_2)$ and $E(G_1 \times_G G_2)$ onto the first or second component (respectively).

Proposition 2.1.1. The graph $G_1 \times_G G_2$ together with the maps π_1, π_2 defined above is the pullback of $G_1 \rightarrow G \leftarrow G_2$ in **Graph**.

Proof. Let Z be any graph and $\psi_1 : Z \rightarrow G_1, \psi_2 : Z \rightarrow G_2$ any morphisms such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$. Define a morphism $\psi : Z \rightarrow G_1 \times_G G_2$ by $\psi_V : z \mapsto (\psi_1(z), \psi_2(z))$. Since $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, $(\psi_{1,V}(z), \psi_{2,V}(z)) \in V(G_1 \times_G G_2)$, so the set map on vertices is well-typed. If $(z, z') \in E(Z)$, then $(\psi_{1,V}(z), \psi_{1,V}(z')) \in E(G_1)$ and $(\psi_{2,V}(z), \psi_{2,V}(z')) \in E(G_2)$, thus $((\psi_{1,V}(z), \psi_{2,V}(z)), (\psi_{1,V}(z'), \psi_{2,V}(z'))) \in E(G_1 \times_G G_2)$. If $(\psi_1(z), \psi_2(z)) = \psi_V(z) = \psi_V(z') = (\psi_1(z'), \psi_2(z'))$, then $\psi_i(z) = \psi_i(z')$ for $i = 1, 2$. As these are injective, $z = z'$, so ψ is indeed a morphism of graphs. It is clear that for $i = 1, 2$, $\pi_{i,V} \circ \psi_V(z) = \psi_{i,V}(z)$, thus $\pi_i \circ \psi = \psi_i$.

Suppose $\tilde{\psi} : Z \rightarrow G_1 \times_G G_2$ is any other map such that $\pi_i \circ \tilde{\psi} = \psi_i$. Since $V(G_1 \times_G G_2) = V(G_1) \times_{V(G)} V(G_2)$, $\tilde{\psi}_V$ is uniquely determined by maps $\tilde{\psi}_{1,V}, \tilde{\psi}_{2,V}$. Since $\pi_{i,V} : V(G_1) \times_{V(G)} V(G_2) \rightarrow V(G_i)$ is clearly the canonical projection map, the condition that $\pi_i \circ \tilde{\psi} = \psi_i$ implies $\tilde{\psi}_{i,V} = \psi_{i,V}$, thus $\tilde{\psi}_V = \psi_V$. Since morphisms of graphs are uniquely determined by how they act on vertices, $\tilde{\psi} = \psi$, so the map defined above is unique. Thus $G_1 \times_G G_2$ satisfies the universal property of the pullback in **Graph**. □

Corollary 2.1.2. The category **Graph** has all pullbacks.

Proof. This is immediate from Proposition 2.1.1. □

Remark 2.1.3. We can be more explicit in our description of the pullback. Since the maps $G_i \hookrightarrow G$ are embeddings, we can identify G_i with its image, thus we can assume the embeddings are actually inclusions. Now a pair $(g_1, g_2) \in V(G_1) \times_{V(G)} V(G_2)$ corresponds to some $g_1 \in G_1, g_2 \in G_2$ such that $g_1 = g_2$. Thus $V(G_1 \times_G G_2) = V(G_1 \cap G_2)$. A similar argument can be made for edges, thus $G_1 \times_G G_2 = G_1 \cap G_2$. Note also that $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$.

2.2 Topologies on Graph

Definition 2.2.1. A collection $\{f_i : G_i \rightarrow G \mid i \in I\}$ in **Graph** is called *jointly surjective* if $\bigcup_{i \in I} f_i(V(G_i)) = V(G)$ and $\bigcup_{i \in I} f_i(E(G_i)) = E(G)$.

Example 2.2.2. If $f : G' \rightarrow G$ is an isomorphism, then $\{f : G' \rightarrow G\}$ is clearly a jointly surjective collection.

Example 2.2.3. Let G be a graph. For each $e = (v_1, v_2) \in E(G)$, let $f_e : P_2 \rightarrow G$ be the map sending the single edge of P_2 to e . Then $\{f_e : P_2 \rightarrow G \mid e \in E(G)\}$ is jointly surjective.

Lemma 2.2.4. If $\{f_i : G_i \rightarrow G \mid i \in I\}$ is jointly surjective, and for each $i \in I$ the collection $\{g_{ij} : G_{ij} \rightarrow G_i \mid j \in J\}$ is jointly surjective, then $\{f_i \circ g_{ij} : G_{ij} \rightarrow G \mid i \in I, j \in J\}$ is jointly surjective.

Proof. Unpacking definitions,

$$\bigcup_{i \in I, j \in J} f_i \circ g_{ij}(V(G_{ij})) = \bigcup_{i \in I} \bigcup_{j \in J} f_i \circ g_{ij}(V(G_{ij})) = \bigcup_{i \in I} f_i \left(\bigcup_{j \in J} g_{ij}(V(G_{ij})) \right) = \bigcup_{i \in I} f_i(V(G_i)) = V(G)$$

The same argument for edges shows $\bigcup_{i \in I, j \in J} f_i \circ g_{ij}(E(G_{ij})) = E(G)$. □

Lemma 2.2.5. If $\{f_i : G_i \rightarrow G\}$ is jointly surjective, for any map $h : G' \rightarrow G$ the collection $\{pr_{2,i} : G_i \times_G G' \rightarrow G' \mid i \in I\}$ is jointly surjective.

Proof. By Remark 2.1.3, we may assume without loss of generality that the f_i are inclusions of subgraphs and that $G_i \times_G G' \rightarrow G'$ is the inclusion $G_i \cap G' \subseteq G'$. In particular, $f_i(S) = S$ for any subset S of $V(G_i)$ or $E(G_i)$. With this in mind,

$$\bigcup_{i \in I} f(V(G_i \cap G')) = \bigcup_{i \in I} V(G_i) \cap V(G') = V(G) \cap V(G') = V(G')$$

The same reasoning holds for E instead of V . □

Corollary 2.2.6. Define a $\mathbf{J}_{\text{jointsurj}}$ -cover as any jointly surjective collection $\{f_i : G_i \rightarrow G \mid i \in I\}$. Then $\mathbf{J}_{\text{jointsurj}}$ is a subgraph topology.

Proof. Example 2.2.2, Lemma 2.2.4, and Lemma 2.2.5 show that the covers in $\mathbf{J}_{\text{edge}}$ have the desired properties. □

Note that every graph G has a $\mathbf{J}_{\text{jointsurj}}$ -covering by its edges (see Example 2.2.3). As will be discussed in greater depth in Section 3 and Section 4, this reflects the fact that this topology will only detect information about the immediate neighbors of a point. Below we list a few other examples of Grothendieck topologies on **Graph**. Verifying these statements is straightforward and left as an exercise to the reader.

Example 2.2.7. If we take coverings of G to be singleton sets containing an isomorphism to G , or the covering in Example 2.2.3, this defines a topology $\mathbf{J}_{\text{edge}}$ on **Graph**.

Example 2.2.8. If we take the coverings of G to be singleton sets containing an isomorphism to G , this defines a subgraph topology $\mathbf{J}_{\text{trivial}}$ on Graph .

Example 2.2.9. If we take coverings of G to be finite jointly surjective collections, this defines a subgraph topology $\mathbf{J}_{\text{fincover}}$ on Graph .

Example 2.2.10. If we take coverings of G to be singleton sets containing an isomorphism to G , or collections $\{f_i : G_i \rightarrow G \mid i \in I\}$ with each G_i finite, this defines a subgraph topology $\mathbf{J}_{\text{fingraph}}$ on Graph .

As discussed in the beginning of this subsection, these subgraph topologies can be used to provide coverings by certain families of graphs. For example, $\mathbf{J}_{\text{edge}}$ corresponds to coverings by P_2 . We now construct a subgraph topology $\mathbf{J}_{\text{cycle}}$ which covers graphs by cycles. This topology will be used in Section 4.

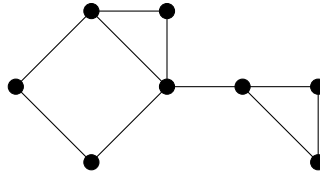
Proposition 2.2.11. There is a subgraph topology $\mathbf{J}_{\text{cycle}}$ whose covers of G are those jointly surjective collections $\{f_i : G_i \rightarrow G \mid i \in I\}$ with the property that for any edge $e \in E(G)$, $f_i^{-1}(e)$ is either empty, or for any cycle C containing e , f_i induces an isomorphism $f_i^{-1}(C) \cong C$.

Proof. It is obvious that if $f : G' \rightarrow G$ is an isomorphism, it is a $\mathbf{J}_{\text{cycle}}$ -cover.

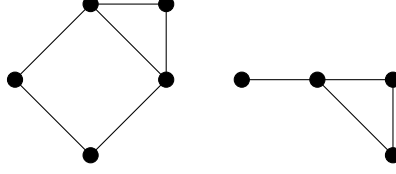
Let $\{f_i : G_i \rightarrow G \mid i \in I\}$ and $\{g_{ij} : G_{ij} \rightarrow G_i \mid j \in J\}$ be $\mathbf{J}_{\text{cycle}}$ -covers of their respective graphs. By Lemma 2.2.4, $\{f_i \circ g_{ij} : G_{ij} \rightarrow G \mid i \in I, j \in J\}$ is jointly surjective. Take $e \in E(G)$ such that $(f_i \circ g_{ij})^{-1}(e) = g_{ij}^{-1}f_i^{-1}(e)$ is nonempty. Take a cycle $C \subseteq G$ containing e . Since $f_i^{-1}(e)$ is nonempty, f_i induces an isomorphism $f_i^{-1}(C) \cong C$. In particular, $f_i^{-1}(C)$ is a cycle in G_i containing $f_i^{-1}(e)$, and since $g_{ij}^{-1}f_i^{-1}(e)$ is nonempty, this means g_{ij} induces an isomorphism $g_{ij}^{-1}f_i^{-1}(C) \cong f_i^{-1}(C)$. Thus $f_i \circ g_{ij}$ induces an isomorphism $(f_i \circ g_{ij})^{-1}(C) \cong C$.

Let $h : G' \rightarrow G$ be any map and $\{f_i : G_i \rightarrow G \mid i \in I\}$ a $\mathbf{J}_{\text{cycle}}$ -cover. By Lemma 2.2.5, $\{f_i \times_G h : G_i \times_G G' \rightarrow G' \mid i \in I\}$ is jointly surjective. We may assume without loss of generality that h and the f_i are inclusions of subgraphs and that $G_i \times_G G' \rightarrow G'$ is the inclusion $G_i \cap G' \subseteq G'$. Take $e \in E(G')$, and let C be a cycle in G' containing e . Then C is a cycle in G , so if $e \in E(G_i)$, $C \subseteq G_i$. But then $C \subseteq G_i \cap G'$, which finishes the proof. $\square$

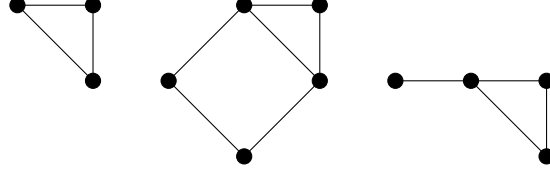
Example 2.2.12. If we interpret the $f_i : G_i \rightarrow G$ in a $\mathbf{J}_{\text{cycle}}$ -cover $\{f_i : G_i \rightarrow G \mid i \in I\}$ as inclusions of subgraphs, then this simply means that if $e \in G_i$, any cycle containing e is contained in G_i . Consider the following graph



Then the following collection of two subgraphs (with their obvious embeddings) would be a $\mathbf{J}_{\text{cycle}}$ -covering



However, the covering shown below would not be a $\mathbf{J}_{\text{cycle}}$ -covering, since the leftmost graph contains the edge contained in both a 3-cycle and 4-cycle, but not the 4-cycle.



Note that Example 2.2.3 is not a $\mathbf{J}_{\text{cycle}}$ -cover for the example above. In fact, it is easily seen it is not a $\mathbf{J}_{\text{cycle}}$ -covering for any graph which contains a cycle.

Remark 2.2.13. Note that all of the results and examples in Sections 2.1 and 2.2 can be done for $\mathbf{Graph}^{\text{fin}}$.

Remark 2.2.14. Due to the author's intentions for this paper, we only consider topologies on undirected. However, there are interesting examples of topologies on directed graphs; two of these are studied extensively in [2].

3 Colorings

This section will introduce the main idea of the paper: colorings of graphs may be viewed as sheaves, and ruled colorings may be viewed as subsheaves. In Section 3.1 we discuss the $\mathbf{Color}^S$ sheaf for a set S . In the next section we finally make precise the notion of a rule. In Section 3.3 we discuss what it means to negate ruled colorings in the internal logic of a topos. In fact, we find an explicit formula for sheaf corresponding to the negation of a rule, see Theorem 3.3.1. Finally, in Section 3.4 we explore the notion of ruled colorings parameterized over some base category.

3.1 Colorings as Sheaves

Let S be any set. Consider the presheaf $\mathbf{Color}^S : \mathbf{Graph}^{\text{op}} \rightarrow \mathbf{Set}$ which acts on objects by $G \mapsto \text{Hom}_{\mathbf{Set}}(V(G), S)$ and morphisms by $[f : G \rightarrow G'] \mapsto [f^* : \phi \mapsto \phi \circ f]$. Note that $\mathbf{Color}^S(G)$ is the set of all S -valued colorings of G . By We will use the notation $\phi|_G := f^*(\phi)$.

Proposition 3.1.1. Let $\mathbf{J}$ be any subgraph topology. Then $\mathbf{Color}^S$ is a sheaf on $(\mathbf{Graph}, \mathbf{J})$.

Proof. Let $\{f_i : G_i \rightarrow G \mid i \in I\}$ be a $\mathbf{J}$ -cover of G . Using the identification of embeddings as subgraphs and pullbacks as intersections, by Remark 5.0.5 it suffices to show that for any collection $(g_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_{\mathbf{Set}}(V(G_i), S)$ such that $g_i|_{V(G_i \cap G_j)} = g_j|_{V(G_i \cap G_j)}$ for all $i, j \in I$, there exists a unique $g \in \text{Hom}_{\mathbf{Set}}(V(G), S)$ such that $g|_{V(G_i)} = g_i$. But this is just a basic fact about set-theoretic functions: since the collection $\{f_i : G_i \hookrightarrow G \mid i \in I\}$ is jointly surjective,

for any $v \in V(G)$ there exists some $i \in I$ such that $v \in G_i$, and we define $g(v) := g_i(v)$. The relation $g_i|_{V(G_i \cap G_j)} = g_j|_{V(G_i \cap G_j)}$ implies that if $v \in V(G_i \cap G_j)$, $g_i(v) = g_j(v)$ is well-defined. Unique is clear. $\square$

This will serve as the “starting sheaf” which we use to define ruled colorings.

3.2 Ruled Colorings

Intuitively speaking, a ruled coloring should be determined by some property (called the rule). A given coloring of a graph may satisfy this rule, or it may satisfy the rule on some subgraphs, or it may satisfy the rule nowhere. Furthermore, this rule should be “the same” across all graphs. With these observations, it is natural to define ruled colorings in the following manner, using the language of Grothendieck topoi.

Let $\mathbf{J}$ be a subgraph topology, $\mathbf{Sh}(\mathbf{Graph}, \mathbf{J})$ the topos of sheaves over $(\mathbf{Graph}, \mathbf{J})$, and Ω the subobject classifier of $\mathbf{Sh}(\mathbf{Graph}, \mathbf{J})$. Inspired by the internal-external viewpoints discussed in 5, we define ruled colorings as follows:

Definition 3.2.1. Externally: a $\mathbf{J}$ -ruled S -coloring is a subsheaf of $\mathbf{Color}^S$ in $\mathbf{Sh}(\mathbf{Graph}, \mathbf{J})$. If S is clear from context, we will call this a $\mathbf{J}$ -ruled coloring.

Definition 3.2.2. Internally: a $\mathbf{J}$ -rule for S -colorings is a morphism $\mathbf{Color}^S \rightarrow \Omega$ in $\mathbf{Sh}(\mathbf{Graph}, \mathbf{J})$. If S is clear from context, we will just call this a $\mathbf{J}$ -rule.

By definition of the subobject classifier, there is a bijection between $\mathbf{J}$ -rules and $\mathbf{J}$ -ruled colorings given by pulling back the $\mathbf{J}$ -rule along the universal map $\text{true} : * \rightarrow \Omega$.

Example 3.2.3. Suppose $T \subseteq S$. Then $\mathbf{Color}^T(G) \subseteq \mathbf{Color}^S(G)$ for any graph G , so we have a monomorphism $\mathbf{Color}^T \hookrightarrow \mathbf{Color}^S$. By the previous sentence, this is the pullback of true along some unique map $\mathbf{Color}^S \rightarrow \Omega$, which we will denote by $T_{\subseteq S}$. Note that if $T = T'$, then $T_{\subseteq S} = T'_{\subseteq S}$.

Lemma 3.2.4. Let true_S denote the composition $\mathbf{Color}^S \rightarrow * \xrightarrow{\text{true}} \Omega$. Then $\mathbf{Color}_{\text{true}_S}^S \cong \mathbf{Color}^S$.

Proof. By the pasting lemma for pullback squares, we can compute the pullback of true_S by computing the following two pullback diagrams

$$\begin{array}{ccccc} \mathbf{Color}_{\text{true}_S}^S & \longrightarrow & P & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \text{true} \\ \mathbf{Color}^S & \longrightarrow & * & \xrightarrow{\text{true}} & \Omega \end{array}$$

Since $*$ is the final object, for any sheaf $\mathcal{F}$ there exists a unique pair of maps $x, y : \mathcal{F} \rightarrow *$ with $\text{true} \circ x = \text{true} \circ y$, thus there is a unique map $\mathcal{F} \rightarrow \mathcal{P}$. Thus P is a final object, so the unique map $P \rightarrow *$ is an isomorphism. The result now follows from the above diagram and the fact that a pullback of an isomorphism is an isomorphism. $\square$

In particular, note that $S_{\subseteq S} = \text{true}_S$. Now we will turn our attention to the logical aspects of this setup. The object Ω comes equipped with the structure of an internal Heyting algebra [3, Chapter IV.8], thus we have “logical operations” $\wedge, \vee, \Rightarrow : \Omega \times \Omega \rightarrow \Omega$ and $\neg : \Omega \rightarrow \Omega$, as well as the “universal truth” map $\text{true} : * \rightarrow \Omega$. These allow us to construct new **J**-rules from existing ones; if ψ, ϕ are **J**-rules, we have

$$\phi \wedge \psi : \mathbf{Color}^S \xrightarrow{\phi \times \psi} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

$$\phi \vee \psi : \mathbf{Color}^S \xrightarrow{\phi \times \psi} \Omega \times \Omega \xrightarrow{\vee} \Omega$$

$$\phi \Rightarrow \psi : \mathbf{Color}^S \xrightarrow{\phi \times \psi} \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

$$\neg \phi : \mathbf{Color}^S \xrightarrow{\phi} \Omega \xrightarrow{\neg} \Omega$$

Since $\langle \Omega, \vee, \wedge, \Rightarrow, \neg \rangle$ the structure of an internal Heyting algebra, any statement of formal logic which is true in a Heyting algebra will be true for the operations above, i.e. the corresponding map will factor through $\text{true} : * \rightarrow \Omega$. For example, $x \Rightarrow \neg \neg x$ is always true in a Heyting algebra **source**, and we will see below that $\phi \Rightarrow \neg \neg \phi$ indeed factors through true for any **J**-rule ϕ . As another example, in a Heyting algebra we have $a \wedge (b \vee c) = (a \vee b) \wedge (a \vee c)$ **source**, thus a $\phi \wedge (\psi \vee \rho)$ -coloring is equivalent to a $(\phi \vee \psi) \wedge (\phi \vee \rho)$ -coloring. We will see shortly that this fits our intuition.

Let G be a graph, $\alpha \in \mathbf{Color}^S(G)$ an S -coloring, and ϕ a **J**-rule. We say that α is a ϕ -coloring if $\alpha \in \mathbf{Color}_\phi^S(G)$. Given a map $f : G' \rightarrow G$, let $\tilde{f}^S : \mathbf{Color}^S(G) \rightarrow \mathbf{Color}^S(G')$ be the map induced by f . We can interpret the logical operations above using sheaf semantics, as discussed in VI.7.1 of [3]:

Lemma 3.2.5. (Theorem VI.7.1 [3]) *Let G be a graph and $\alpha \in \mathbf{Color}^S(G)$ an S -coloring. Let ϕ, ψ be **J**-rules. Then the following hold:*

1. α is a $\phi \wedge \psi$ -coloring iff α is a ϕ -coloring and α is a ψ -coloring.
2. α is a $\phi \vee \psi$ -coloring iff there exists a **J**-cover $\{f_i : G_i \rightarrow G \mid i \in I\}$ such that for each $i \in I$, $\tilde{f}_i^S(\alpha)$ is either a ϕ -coloring or a ψ -coloring on G_i .
3. α is a $\phi \Rightarrow \psi$ coloring iff α being a ϕ -coloring implies α is a ψ -coloring.
4. α is a $\neg \phi$ coloring iff for all $f : G' \rightarrow G$, if $\tilde{f}^S(\alpha)$ is a ϕ -coloring on G' then f is an empty cover.
5. *Monotonicity:* Given a map $f : G' \rightarrow G$, if α is a ϕ -coloring on G , then $\tilde{f}^S(\alpha)$ is a ϕ -coloring on G' .
6. *Local character:* Let $\alpha \in \mathbf{Color}^S(G)$. If $\{f_i : G_i \rightarrow G \mid i \in I\}$ is a **J**-cover such that for each $i \in I$, $\tilde{f}_i^S(\alpha)$ is a ϕ -coloring on each G_i , then α is a ϕ -coloring on G .

Remark 3.2.6. Since we may view maps in **Graph** as embeddings of subgraphs, we adopt the suggestive notation $\alpha|_{G_i} := \tilde{f}_i^S(\alpha)$, which we read as “ α restricted to G_i .”

Returning to our earlier example, α is a $\phi \wedge (\psi \vee \rho)$ -coloring iff it is a ρ -coloring and there exists a **J**-cover $\{G_i \hookrightarrow G \mid i \in I\}$ such that $\alpha|_{G_i}$ is either a ψ -coloring or a ρ -coloring for each $i \in I$. But this is evidently the same as saying that for each $i \in I$, $\alpha|_{G_i}$ is either a ϕ coloring and a ψ coloring, or a ϕ coloring and a ρ coloring. Repacking everything, this is saying that α is a $(\phi \wedge \psi) \vee (\phi \vee \rho)$ -coloring, which explicitly verifies our intuition and the earlier statement of de Morgan's law.

3.3 Negating Rules

Now we turn our attention to negation. Using Remark 3.2.6 with Lemma 3.2.5(4), we may rephrase α being a $\neg\phi$ -coloring as saying that if for some subgraph G' of G , if $\alpha|_{G'}$ is a ϕ -coloring, then $G' \hookrightarrow G$ is the empty cover. But since subgraph topologies have only jointly surjective covers, the only object having the empty cover is the empty graph itself. Thus $G = \emptyset$. In other words, α is a $\neg\phi$ -coloring if there are no subgraphs of G on which the restriction of α is a ϕ -coloring. But note that each vertex of G is a subgraph (with embedding a map of the form $* \rightarrow G$, thus α restricted to each vertex is not a ϕ -coloring. In other words, $\alpha|_g \notin \mathbf{Color}_\phi^S(*)$ for each $g \in V(G)$. But $\mathbf{Color}_\phi^S(*) = \mathbf{Hom}_{\mathbf{Set}}(V(*), S) \cong S$, the isomorphism being given by evaluation, thus $\alpha|_g$ may be identified with $\alpha(g)$. Since $\mathbf{Color}_\phi^S$ is a subsheaf of $\mathbf{Color}^S$, we have an inclusion $\mathbf{Color}_\phi^S(*) \subseteq \mathbf{Color}^S(*)$, which we may identify with some subset $S_\phi \subseteq S$. Putting this together, we see that if α is a $\neg\phi$ coloring, then $\alpha(g) \notin S_\phi$ for all $g \in V(G)$, thus α is a $(S \setminus S_\phi)_{\subseteq S}$ -coloring. Furthermore, suppose $\alpha(g) \notin S_\phi$ for all $g \in V(G)$. Let G' be any subgraph of G , and suppose $\alpha|_{G'}$ is a ϕ -coloring. For any vertex $g \in V(G')$ identified with a map $* \hookrightarrow G'$, the restriction map $\mathbf{Color}_\phi^S(G') \rightarrow \mathbf{Color}_\phi^S(*)$ implies $\alpha|_g \in \mathbf{Color}_\phi^S(*)$, i.e. $\alpha(g) \in S_\phi$. But this contradicts our earlier assumption, thus $\alpha|_{G'} \notin \mathbf{Color}_\phi^S(G')$ for any subgraph G' of G , i.e. α is a $\neg\phi$ -coloring on G . As this holds for any graph G , we conclude

Theorem 3.3.1. *Let S be a set, ϕ be a **J**-rule for S -colorings, and S_ϕ as denoted above. Then $\mathbf{Color}^{S \setminus S_\phi} = \mathbf{Color}_{\neg\phi}^S$. Internally, this means $(S \setminus S_\phi)_{\subseteq S} = \neg\phi$.*

Corollary 3.3.2. *Let $T \subseteq S$. Then $\neg T_{\subseteq S} = (S \setminus T)_{\subseteq S}$.*

Proof. This follows from Theorem 3.3.1 and the observation that $\mathbf{Color}_{T_{\subseteq S}}^S(*) = \mathbf{Color}^T(*) \cong T$. □

Corollary 3.3.3. *Let $T \subseteq S$. Then $\neg\neg T_{\subseteq S} = T_{\subseteq S}$.*

Proof. By Corollary 3.3.2, $\neg\neg T_{\subseteq S} = \neg(S \setminus T)_{\subseteq S} = (S \setminus (S \setminus T))_{\subseteq S} = T_{\subseteq S}$. □

Corollary 3.3.4. *Let ϕ be any **J**-rule. Then $\neg\neg\neg\phi = \neg\phi$.*

Proof. By Theorem 3.3.1 and Corollary 3.3.3, $\neg\neg\neg\phi = \neg\neg(S \setminus S_\phi)_{\subseteq S} = (S \setminus S_\phi)_{\subseteq S} = \neg\phi$. □

We will soon see that the statement $\neg\neg\phi = \phi$ is not true in general, i.e. the internal logic is not Boolean. It is easy to see, however, that $\mathbf{Color}_\phi^S$ is a subobject of $\mathbf{Color}^{S \setminus S_\phi}$ for any ϕ , which by Lemma 3.2.5(3) shows $\phi \Rightarrow \neg\neg\phi$ is universally valid (i.e. is the entire sheaf $\mathbf{Color}^S$).

3.4 Parameterized Colorings

By [5, 1. Definition], there is a fully faithful functor $\text{Sh}(\text{Graph}, \mathbf{J}) \hookrightarrow \text{Fun}(\text{Graph}^{\text{op}}, \text{Set})$ which admits a left exact adjoint. Applying $\text{Fun}(\mathbf{C}, -)$ to both sides and using [4, 3. Properties, Pre- and composite adjoint functors], we get a left adjoint to $\text{Fun}(\mathbf{C}, \text{Sh}(\text{Graph}, \mathbf{J})) \hookrightarrow \text{Fun}(\mathbf{C}, \text{Fun}(\text{Graph}^{\text{op}}, \text{Set}))$. Since limits in functor categories are computed pointwise, the left adjoint is left exact. Since $\text{Fun}(\mathbf{C}, \text{Fun}(\text{Graph}^{\text{op}}, \text{Set})) \cong \text{Fun}(\mathbf{C} \times \text{Graph}^{\text{op}}, \text{Set})$, [5, 1. Definition] implies $\text{Fun}(\mathbf{C}, \text{Sh}(\text{Graph}, \mathbf{J}))$ is a topos. If $\mathbf{C}$ has a functor $U : \mathbf{C} \rightarrow \text{Set}$, then the following defines a functor (which we will call **Color**)

$$A \in \text{Ob}(\mathbf{C}) \mapsto \mathbf{Color}^{U(A)}$$

$$[A \xrightarrow{f} B] \in \text{Mor}(\mathbf{C}) \mapsto [\mathbf{Color}^{U(A)} \xrightarrow{f_*} \mathbf{Color}^{U(B)}]$$

Definition 3.4.1. Let $\mathbf{C}$ be a category equipped with a faithful functor $U : \mathbf{C} \rightarrow \text{Set}$. A $\mathbf{C}$ -parameterized ruled coloring is a subfunctor of **Color**. As in Definitions 3.2.1, 3.2.2, this is equivalently a map $\phi : \mathbf{Color} \rightarrow \tilde{\Omega}$, where $\tilde{\Omega}$ is the subobject classifier of $\text{Fun}(\mathbf{C}, \text{Sh}(\text{Graph}, \mathbf{J}))$.

Remark 3.4.2. To justify the use of the word “parameterized” in the above definition, for any object $A \in \text{Ob}(\mathbf{C})$ there is a functor $\text{ev}_A : \text{Fun}(\mathbf{C}, \text{Sh}(\text{Graph}, \mathbf{J})) \rightarrow \text{Sh}(\text{Graph}, \mathbf{J})$ sending a functor F to $F(A)$. Note that $\text{ev}_A(\mathbf{Color}) = \mathbf{Color}^{U(A)}$. Furthermore, if $\mathbf{Color}' \hookrightarrow \mathbf{Color}$ is a subobject, then by the characterization of monomorphisms in [7] there is a pullback diagram

$$\begin{array}{ccc} \mathbf{Color}' & \xrightarrow{=} & \mathbf{Color}' \\ \downarrow = & \lrcorner & \downarrow \\ \mathbf{Color}' & \hookrightarrow & \mathbf{Color} \end{array}$$

Since limits of functors are given pointwise [6, Proposition 2.1], ev_A preserves this pullback square, thus $\mathbf{Color}'(A)$ is a subfunctor of $\mathbf{Color}^{U(A)}$ (again by [7]).

Example 3.4.3. Let $U : \mathbf{C} \rightarrow \text{Set}$ be as above. For a fixed $A \in \text{Ob}(\mathbf{C})$, consider the assignment $\mathbf{Color}_{\text{monochrome}}^A(G) := \{V(G) \xrightarrow{h} A \mid \exists a \in U(A) \text{ with } h(G) \equiv a\}$ (i.e. constant colorings). If $\iota : G' \hookrightarrow G$ is an embedding, then $h \circ \iota(G') \subseteq h(G) \equiv a$, thus $\iota^*h \in \mathbf{Color}_{\text{monochrome}}^A(G')$. By Remark 5.0.5, it is easy to see this satisfies the gluing condition for any subgraph topology. Thus $\mathbf{Color}_{\text{monochrome}}^A$ is a sheaf. Now suppose we have $f : A \rightarrow B$. Since $U(f)$ sends singletons to singletons, it is clear there is a functor $f_* : \mathbf{Color}_{\text{monochrome}}^A \rightarrow \mathbf{Color}_{\text{monochrome}}^B$. Finally, note that each $\mathbf{Color}_{\text{monochrome}}^A$ is a subsheaf of $\mathbf{Color}^{U(A)}$, thus $\mathbf{Color}_{\text{monochrome}}^{(-)}$ is a subsheaf of **Color**, i.e. a parameterized ruled coloring.

Example 3.4.3 illustrates the guiding principle behind parameterized ruled colorings: these are rules which can be given to colorings valued in a subcollection of sets. We will see another example of this in Proposition 4.2.1.

4 Combinatorial Problems

In this section we begin by discussing how proper coloring problems fit into this framework. After this, we provide an overview of colorings valued in abelian groups and discuss how cohomological tools could be used to count colorings of a graph.

4.1 Proper Colorings

Given a set S , a proper S -coloring of a graph G is a function $f : V(G) \rightarrow S$ with the property that if $f(g_1) = f(g_2)$ with $g_1 \neq g_2$, then there is no edge between g_1 and g_2 . Now take a morphism $h : G_1 \rightarrow G_2$ and proper coloring $f : G_2 \rightarrow \mathbb{N}$. We claim that $f \circ h$ is a proper coloring on G_1 . Suppose $f(h(g_1)) = f(h(g_2))$. Then $h(g_1)$ and $h(g_2)$ do not share an edge, or $h(g_1) = h(g_2)$. But since we assume h is an embedding, the latter cannot hold if $g_1 \neq g_2$, thus $h(g_1)$ and $h(g_2)$ do not share an edge. Consequently, there is a subpresheaf $\mathbf{PropColor}^S : \mathbf{Graph}^{\text{op}} \rightarrow \mathbf{Set}$ of $\mathbf{Color}$ which sends a graph G to the set of proper S -colorings on G .

Proposition 4.1.1. Let $\mathbf{J}$ be a subgraph topology. Then $\mathbf{PropColor}^S$ is a sheaf on $(\mathbf{Graph}, \mathbf{J})$.

Proof. Since $\mathbf{PropColor}^S$ is a subpresheaf of $\mathbf{Color}$, to show it is a sheaf it suffices to show that given any jointly surjective family $\{\iota_i : G_i \rightarrow G \mid i \in I\}$ and proper S -colorings $f_i : G_i \rightarrow S$, the map f given in Proposition 3.1.1 is proper. Indeed, fix a vertex $v \in V(G)$. For any neighbor u of v , the edge (u, v) is contained in some G_i . Since $f|_{G_i} = f_i$ and since this map is proper, $f(u) = f_i(u) \neq f_i(v) = f(v)$. As this holds for any vertex v and any neighbor of v , f is a proper coloring. □

An infamous problem in combinatorics is the following: given a graph G , how many proper k -colorings exist? Using the framework developed earlier, this is simply the question: what is the cardinality of $\mathbf{PropColor}^{\{1, \dots, k\}}(G)$? In general, this tends to be a very difficult question to answer. Using the equalizer property of sheaves, we can place an upper bound on this number, provided we can answer (or at least bound) this question for a cover of the graph. In fact, we can do this for any $\mathbf{J}$ -rule.

Proposition 4.1.2. Let G be a finite graph, S a finite set, and ϕ a $\mathbf{J}$ -rule for S -colorings. Let $\{f_i : G_i \rightarrow S \mid i \in I\}$ be a finite $\mathbf{J}$ -cover of G . Then

$$|\mathbf{Color}_\phi^S(G)| \leq \prod_{i \in I} |\mathbf{Color}_\phi^S(G_i)|$$

Proof. We have an equalizer diagram

$$\mathbf{Color}_\phi^S(G) \longrightarrow \prod_{i \in I} \mathbf{Color}_\phi^S(G_i) \rightrightarrows \prod_{i, j \in I} \mathbf{Color}_\phi^S(G_i \times_G G_j)$$

By basic properties of equalizers, the first map is injective. Taking cardinalities, this implies $|\mathbf{Color}_\phi^S(G)| \leq |\prod_{i \in I} \mathbf{Color}_\phi^S(G_i)| = \prod_{i \in I} |\mathbf{Color}_\phi^S(G_i)|$. □

Shifting topics by continuing the discussion on negations in Section 3.3, **PropColor**^S provides an example of a rule whose double negation is not equal to itself:

Example 4.1.3. Let $\mathbf{p} : \mathbf{Color}^S \rightarrow \Omega$ be the rule associated to **PropColor**^S. By Theorem 3.3.1, $\neg \mathbf{p} = (S \setminus S_{\mathbf{p}})_{\subseteq S}$. Since any coloring of the one-vertex graph is vacuously a proper coloring, $S_{\mathbf{p}} = S$, thus $\neg \mathbf{p} = \emptyset$. By Corollary 3.3.2, $\neg \neg \mathbf{p} = S_{\subseteq S}$. By Corollary 3.3.2 and the comment after Lemma 3.2.4, we see $\neg \neg \mathbf{p} = \text{true}_S \neq \mathbf{p}$.

4.2 Colorings Valued in Abelian Groups

Let A be an abelian group with identity 0. Then **Color**^A admits the structure of an abelian group. Indeed, if $\phi, \psi : V(G) \rightarrow A$ are set-theoretic functions, then pointwise addition $(\phi + \psi)(a) := \phi(a) + \psi(a)$ defines a group structure on $\text{Hom}_{\text{Set}}(V(G), A)$, where the identity element is the constant map $V(G) \mapsto 0$. We can then look at group-ruled colorings valued in abelian groups, where a group-rule for a subgraph topology $\mathbf{J}$ is a $\mathbf{J}$ -rule such that the corresponding subsheaf is a subsheaf of abelian groups (not just of sets). In this case, we can pose a similar question to that discussed in 4.1: given a group-rule ϕ and graph G , what is the cardinality of **Color** _{ϕ} ^A(G), i.e. how many ϕ -colorings of G exist? By using abelian-group valued colorings, we can apply tools from sheaf cohomology to compute the size and elements of **Color** _{ϕ} ^A(G). In particular, [8, Lemma 19.7.4] states that for any subgraph topology $\mathbf{J}$, $(\mathbf{Graph}, \mathbf{J})$ has enough injectives, thus for any graph G the right derived functors of $\mathcal{F} \mapsto \mathcal{F}(G)$ exist (where $\mathcal{F}$ is a sheaf of abelian groups over $(\mathbf{Graph}, \mathbf{J})$). In particular, we can consider the case where $\mathcal{F} = \mathbf{Color}_{\phi}^A$, in which case we denote the i -th cohomology group by $H^i(G, A, \phi)$. Moreover, if we have some exact sequence of group-ruled colorings $0 \rightarrow \mathbf{Color}_{\phi}^A \rightarrow \mathbf{Color}_{\psi}^B \rightarrow \mathbf{Color}_{\rho}^C \rightarrow 0$, then for any graph G we have a long exact sequence

$$0 \rightarrow H^0(G, A, \phi) \rightarrow H^0(G, B, \psi) \rightarrow H^0(G, C, \rho) \rightarrow H^1(G, A, \phi) \rightarrow \dots$$

Since $H^0(G, A, \phi) = \mathbf{Color}_{\phi}^A(G)$, this provides us with a way of using cohomological methods to answer the question posed earlier.

Definition 4.2.1. Given an abelian group A , a *cyclic additive A-coloring* is a coloring $\phi : V(G) \rightarrow A$ with the property that for any cycle $C \subseteq G$, $\sum_{g \in V(C)} \phi(g) = 0$. We denote the set of such colorings **Color**_{add}^A(G).

Proposition 4.2.2. The functor **Color**_{add} is a subsheaf of **Color** : $\mathbf{Ab}^{\text{fin}} \rightarrow \text{Sh}_{\mathbf{Ab}^{\text{fin}}}(\mathbf{Graph}^{\text{fin}}, \mathbf{J}_{\text{cycle}})$.

Proof. Since any group homomorphism $\psi : A \rightarrow B$ preserves 0, it is clear that $\psi_* : \mathbf{Color}_{\text{add}}^A(G) \rightarrow \mathbf{Color}_{\text{add}}^B(G)$ is well-typed for any graph G . Since a cycle in a subgraph G of G' is a cycle in G' , it is clear that $\iota : G \hookrightarrow G'$ induces by precomposition a map $\mathbf{Color}_{\text{add}}^A(G') \rightarrow \mathbf{Color}_{\text{add}}^A(G)$. Since pre and post composition commute, it is clear that for any $G \hookrightarrow G'$, the following square commutes,

$$\begin{array}{ccc} \mathbf{Color}_{\text{add}}^A(G') & \xrightarrow{\psi_*} & \mathbf{Color}_{\text{add}}^B(G') \\ \iota^* \downarrow & & \downarrow \iota^* \\ \mathbf{Color}_{\text{add}}^A(G) & \xrightarrow{\psi_*} & \mathbf{Color}_{\text{add}}^B(G) \end{array}$$

thus the assignment $A \mapsto \mathbf{Color}_{\text{add}}^A, \psi \mapsto \psi_*$ determines a functor $\mathbf{Ab}^{\text{fin}} \rightarrow \mathbf{PSh}_{\mathbf{Ab}^{\text{fin}}}(\mathbf{Graph}^{\text{fin}})$. We also see that $\mathbf{Color}_{\text{add}}^A$ is a subpresheaf of $\mathbf{Color}^A$ for every abelian group A , thus $\mathbf{Color}_{\text{add}}$ is a subpresheaf of $\mathbf{Color}$. It remains to show then that $\mathbf{Color}_{\text{add}}^A$ is a sheaf on the cycle topology.

Since limits in finite abelian groups agree with limits in set (the functor $\mathbf{Ab}^{\text{fin}} \hookrightarrow \mathbf{Ab}$ can be checked to preserve limits, and $\mathbf{Ab} \rightarrow \mathbf{Set}$ admits a left adjoint, thus preserves limits), by Remark 5.0.5 it suffices to check that given a cover $\{\iota_i : G_i \rightarrow G \mid i \in I\}$ be a cover of a graph G in the cycle topology, an abelian group A , and cyclic additive A -coverings $\psi_i : V(G_i) \rightarrow A$ such that $\psi_i|_{G_i \cap G_j} = \psi_j|_{G_i \cap G_j}$, the map $\psi : V(G) \rightarrow A$ from Proposition 3.1.1 is cyclic additive. Pick a cycle $C_n \subseteq G$. By definition of the cycle topology, $C_n \subseteq G_i$ for some $i \in I$. Since $\psi|_{G_i} = \psi_i$ and since ψ_i is cyclic additive, it follows that $\sum_{v \in C_n} \psi(v) = 0$. Since C_n is arbitrary, ψ must be cyclic additive. $\square$

In future work, we hope to apply the aforementioned cohomological tools to count the number of cyclic additive colorings of a graph.

5 Appendix: Review of Grothendieck Topologies and their Sheaves

In an effort to keep this paper as self-contained as is reasonable, we include a few basic definitions and a result from the theory of sites as a reference. A comprehensive discussion can be found in Chapters III and IV of [3].

Definition 5.0.1. (Definition III.2.2 [3]) A Grothendieck topology on a category $\mathbf{C}$ with pullbacks is a collection $\mathbf{J}$ consisting of (for each object C of $\mathbf{C}$) a collection of sets of the form $\{f_i : C_i \rightarrow C \mid i \in I\}$ (called “covers” of C) satisfying the following three properties:

1. If $f : C' \rightarrow C$ is an isomorphism, then $\{f : C' \rightarrow C\}$ is a cover.
2. If $\{f_i : C_i \rightarrow C \mid i \in I\}$ is a cover for C , then for any $g : D \rightarrow C$, $\{\text{pr}_{2,i} : C_i \times_C D \rightarrow D \mid i \in I\}$ is a cover for D .
3. If $\{f_i : C_i \rightarrow C \mid i \in I\}$ is a cover for C , and $\{g_{ij} : C_{ij} \rightarrow C_i \mid j \in J_i\}$ is a cover for each $i \in I$, then $\{f_i \circ g_{ij} : C_{ij} \rightarrow C \mid i \in I, j \in J_i\}$ is a cover of C .

Remark 5.0.2. The definition given above is technically the definition of a “basis for a Grothendieck topology,” and a Grothendieck topology is defined in terms of objects called sieves (Definition III.2.1 [3]). However, we will never use the latter description in this paper, hence the notation in the above definition.

Definition 5.0.3. A site is a pair $(\mathbf{C}, \mathbf{J})$ with $\mathbf{C}$ a category with pullbacks and $\mathbf{J}$ a Grothendieck topology on $\mathbf{C}$.

Definition 5.0.4. (Proposition III.2.1 [3]) A sheaf on a site $(\mathbf{C}, \mathbf{J})$ is a presheaf $\mathcal{F}$ (i.e. functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$) such that for each object C of $\mathbf{C}$ and each cover $\{f_i : C_i \rightarrow C \mid i \in I\}$, the following is an equalizer:

$$\mathcal{F}(C) \longrightarrow \prod_{i \in I} \mathcal{F}(C_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(C_i \times_C C_j)$$

Here the top map is the one induced by the projection onto the first argument (recall that $\mathcal{F}$ is contravariant), and the bottom map is the one induced by projection onto the second argument.

A morphism of sheaves is a natural transformation of the underlying functors. We denote the category of sheaves by $\text{Sh}(\mathbf{C}, \mathbf{J})$ and call such a category a Grothendieck topos.

Remark 5.0.5. Note that since every set is a disjoint union of points, it suffices to check the universal property for maps $*$ $\rightarrow$ $\mathcal{F}(C_i)$. In this case, the sheaf condition becomes the statement that for any cover $\{f_i : C_i \rightarrow C \mid i \in I\}$, if there is a collection $(\alpha_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(C_i)$ such that $(\alpha_j)|_{C_i \times_C C_j} = (\alpha_i)|_{C_i \times_C C_j}$ for each pair i, j , there exists an $\alpha \in \mathcal{F}(C)$ such that $\text{res}_{C_i}^C(\alpha) = \alpha_i$ for each $i \in I$.

Definition 5.0.6. A subobject classifier of a site $\text{Sh}(\mathbf{C}, \mathbf{J})$ is a sheaf Ω such that $\text{Sub}(-) = \text{Hom}_{\text{Sh}(\mathbf{C}, \mathbf{J})}(-, \Omega)$, where $\text{Sub}(F)$ is the class of subobjects of F , i.e. isomorphisms of monomorphism classes under F .

Proposition 5.0.7. [3, Proposition III.7.3] Every site has a subobject classifier.

We omit the proof and instead refer the reader to the included reference.

References

- [1] Ingo Blechschmidt. Using the internal language of toposes in algebraic geometry, 2021.
- [2] Joel Friedman. Sheaves on graphs, their homological invariants, and a proof of the hanna neumann conjecture, 2011.
- [3] Saunders Mac Lane. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext Ser. Springer New York, New York, NY, 1st ed edition, 1994.
- [4] nLab authors. adjoint functor. <https://ncatlab.org/nlab/show/adjoint+functor>, October 2026. Revision 156.
- [5] nLab authors. Grothendieck topos. <https://ncatlab.org/nlab/show/Grothendieck+topos>, October 2026. Revision 67.
- [6] nLab authors. limits of presheaves are computed objectwise. <https://ncatlab.org/nlab/show/limits+of+presheaves+are+computed+objectwise>, October 2026. Revision 6.
- [7] nLab authors. monomorphism. <https://ncatlab.org/nlab/show/monomorphism>, September 2026. Revision 48.
- [8] The Stacks project authors. The stacks project. <https://stacks.math.columbia.edu>, 2026.