

Logic in Topos Theory

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These first part of these notes outline the main results in *Sheaves in Geometry and Logic* (Mac Lane and Moerdijk) IV.7, VI.5, VI.6, and VI.7.

Heyting Algebras in Topoi

Heyting Algebra

A Heyting algebra is a poset with all finite products ($\wedge$), coproducts ($\vee$), a final object (1), an initial object (0), and all exponentials ($\Rightarrow$).

Example 1 *(The set of subobjects $\text{Sub}_{\mathcal{E}}(1)$ of a topos $\mathcal{E}$ is a Heyting algebra.)*

For subobjects $A, B \in \text{Sub}_{\mathcal{E}}(1)$, $A \leq B$ iff there exists a monic $A \rightarrowtail B$. The product of two subobjects is given by intersection $A \cap B$, the coproduct by union $A \cup B$, exponentials by B^A , initial object by $0 \rightarrowtail 1$, and terminal object by $1 \rightarrowtail 1$.

Take two maps $a, b : T \rightarrow B^A$. Associated to these are maps $\tilde{a}, \tilde{b} : A \times T \rightarrow B$. If $B \rightarrowtail 1$, then $\tilde{a} = \tilde{b}$, thus $a = b$, so $B^A \rightarrowtail 1$ is monic.

Any map in $\mathcal{E}$ of subobjects A, B of 1 is monic since $A \rightarrow B \rightarrowtail 1 = A \rightarrowtail 1$. Since $A \cap B$ is pullback along the unique maps $A \rightarrowtail 1, B \rightarrowtail 1$, $A \cap B = A \times B$. Therefore

$$\text{Hom}_{\text{Sub}_{\mathcal{E}}(1)}(C \cap A, B) = \text{Hom}_{\mathcal{E}}(C \times A, B) = \text{Hom}_{\mathcal{E}}(C, B^A) = \text{Hom}_{\text{Sub}_{\mathcal{E}}(1)}(C, B^A)$$

Internal Heyting Algebra

Given a category $\mathcal{C}$ with finite limits, an internal lattice L is an object of $\mathcal{C}$ with maps $\wedge : L \times L \rightarrow L$, $\vee : L \times L \rightarrow L$, $\top : 1 \rightarrow L$, $\perp : 1 \rightarrow L$, $\Rightarrow : L \times L \rightarrow L$ satisfying the diagrammatic versions of the following:

$$x \wedge y = y \wedge x, \quad x \vee y = y \vee x$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z$$

$$x \wedge x = x, \quad x \vee x = x$$

$$x \wedge (y \vee x) = x = (x \vee y) \wedge x$$

$$x \vee \perp = x, \quad x \wedge \top = x$$

$$(x \Rightarrow x) = \top$$

$$x \wedge (x \Rightarrow y) = x \wedge y, \quad y \wedge (x \Rightarrow y) = y$$

$$x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$$

Example 2 *(The object classifier Ω of a topos $\mathcal{E}$ is an internal Heyting algebra)*

The map $!_A^{-1} : \text{Sub}_{\mathcal{E}/A}(1) \rightarrow \text{Sub}_{\mathcal{E}}(A)$ induced by the unique map $!_A : A \rightarrow 1$ is an isomorphism such that $!_A^* : \mathcal{E} \rightarrow \mathcal{E}/1$ preserves all topos structure, so $\text{Sub}_{\mathcal{E}}(A)$ carries a Heyting algebra structure in the analogous way. Passing to slices, for a map $A \rightarrow B$ we get a morphism of Heyting algebras $\text{Sub}_{\mathcal{E}}(B) \rightarrow \text{Sub}_{\mathcal{E}}(A)$. Thus $\text{Sub}_{\mathcal{E}}(X) \cong \text{Hom}_{\mathcal{E}}(X, \Omega)$ carries a Heyting algebra structure natural in X , so by Yoneda Ω carries the structure of an internal Heyting algebra.

Given a Heyting algebra L , there is a “negation” map $\neg : L \rightarrow L$ sending $\neg x = (x \Rightarrow 0)$.

Mitchell-Bénabou Language

Fix a topos $\mathcal{E}$ with subobject classifier Ω . The types are objects of the topos. A **term** is a symbol, while its **interpretation** is an arrow. A term σ of type X (denoted $\sigma : X$) refers to a term whose interpretation ends in X .

1. A **variable** x is a term of type X interpreted by $\text{id} : X \rightarrow X$
2. Given terms $\sigma : U \rightarrow X, \tau : V \rightarrow Y$, there is a term $\langle \sigma, \tau \rangle$ of type $X \times Y$ interpreted by $\langle \sigma\pi_U, \tau\pi_V \rangle : U \times V \rightarrow X \times Y$
3. Given terms $\sigma : U \rightarrow X, \tau : V \rightarrow X$, there is a term $\sigma = \tau$ of type Ω with interpretation

$$(\sigma = \tau) : U \times V \xrightarrow{\langle \sigma\pi_U, \tau\pi_V \rangle} X \times X \xrightarrow{\delta_X} \Omega$$

4. Given an arrow $f : X \rightarrow Y$ of $\mathcal{E}$ and term $\sigma : U \rightarrow X$, there is a term $f \circ \sigma : Y$ with interpretation

$$f \circ \sigma : U \xrightarrow{\sigma} X \xrightarrow{f} Y$$

5. Given terms $\theta : V \rightarrow Y^X, \sigma : U \rightarrow X$, there is a term $\theta(\sigma) : Y$ with interpretation

$$\theta(\sigma) : U \times V \xrightarrow{\langle \theta\pi_V, \sigma\pi_U \rangle} Y^X \times X \xrightarrow{e} Y$$

6. Given terms $\sigma : U \rightarrow X, \tau : V \rightarrow \Omega^X$, there is a term $(\sigma \in \tau) : \Omega$ with interpretation

$$\sigma \in \tau : U \times V \xrightarrow{\langle \sigma\pi_U, \tau\pi_V \rangle} X \times \Omega^X \xrightarrow{e} \Omega$$

7. Given variable $x : X$ and term $\sigma : X \times U \rightarrow Z$, there is a term $\lambda x \sigma : Z^X$ with interpretation the transpose of σ .

Terms of type Ω are called **formulas**. We can combine formulas using the internal Heyting algebra structure on Ω . Let $\phi : U \rightarrow \Omega, \psi : V \rightarrow \Omega$ be formulas. Then we have

$$\phi \wedge \psi : U \times V \rightarrow \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

$$\phi \vee \psi : U \times V \rightarrow \Omega \times \Omega \xrightarrow{\vee} \Omega$$

$$\phi \Rightarrow \psi : U \times V \rightarrow \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

$$\neg \phi : U \rightarrow \Omega \xrightarrow{\neg} \Omega$$

We can also interpret quantifiers. The notation $\phi(x, y)$ will be used to denote a formula whose interpretation has source $X \times Y$, where $x : X, y : Y$ are variables. Let $!_X : X \rightarrow 1$ be the terminal map. There is an induced map $P!_X : P1 = \Omega \rightarrow PX = \Omega^X$ with internal adjoints $\forall_X : \Omega^X \rightarrow \Omega$ (right) and $\exists_X : \Omega^X \rightarrow \Omega$ (left). The notation $\forall x \phi(x, y)$ is shorthand for $\forall_X \circ \lambda x \phi(x, y)$.

Given a formula $\phi(x, y)$, there is a subobject $\{(x, y) \mid \phi(x, y)\} \rightarrowtail X \times Y$ defined as the pullback of $\phi(x, y)$ along $\text{true} : 1 \rightarrow \Omega$. A formula $\phi(x, y)$ is **universally valid in $\mathcal{E}$** iff $\phi(x, y) : X \times Y \rightarrow \Omega$ factors through $1 \rightarrow \Omega$ iff $\{(x, y) \mid \phi(x, y)\} \rightarrowtail X \times Y$ is $X \times Y$ itself.

Joyal-Kripke Semantics

We want a way to turn things into human language. Let $\alpha : U \rightarrow X$ be a generalized element. Given a formula $\phi(x) : X \rightarrow \Omega$, we say **U forces $\phi(\alpha)$, or $U \Vdash \phi(\alpha)$** iff $\alpha : U \rightarrow X$ factors through $\{x \mid \phi(x)\} \rightarrowtail X$. The generalization to multiple free variables is straightforward: given $\alpha_1 :$

$U \rightarrow X_1, \dots, \alpha_n : U \rightarrow X_n$ and a formula $\phi(x_1, \dots, x_n)$, we say that $U \Vdash \phi(\alpha_1, \dots, \alpha_n)$ iff the map $\langle \alpha_1, \dots, \alpha_n \rangle : U \rightarrow X_1 \times \dots \times X_n$ factors through $\{(x_1, \dots, x_n) \mid \phi(x_1, \dots, x_n)\}$. If $\phi : 1 \rightarrow \Omega$, then $1 \Vdash \phi$ iff $\phi = \text{true}$.

Simple Example 3 (*Forcing equality*)

Given $\alpha, \beta : U \rightarrow X$, let $\alpha = \beta : U \rightarrow X$ be the composite $U \xrightarrow{\langle \alpha, \beta \rangle} X \times X \xrightarrow{\delta_X} \Omega$. Here $\delta_X(x, x)$ is the formula, and $\langle \alpha, \beta \rangle$ the generalized element. Then

$$U \Vdash \alpha = \beta \iff \alpha = \beta \text{ factors through } \Delta \iff (\alpha = \beta) = \text{true} \circ !_X$$

This is seen in the diagram below:

$$\begin{array}{ccccc} & & X & \xrightarrow{!_X} & 1 \\ & \nearrow & \downarrow \Delta_X & & \downarrow \text{true} \\ U & \xrightarrow{\langle \alpha, \beta \rangle} & X \times X & \xrightarrow{\delta_X} & \Omega \end{array}$$

Proposition 1 (*Monotonicity and local character*)

Let $\alpha : U \rightarrow X$ be a generalized element and $\phi(x)$ a formula. Then the following hold:

Monotonicity: If $U \Vdash \phi(\alpha)$, then for any $f : U' \rightarrow U$ in $\mathcal{E}$, we have $U' \Vdash \phi(\alpha \circ f)$

Local character: If $f : U' \rightarrow U$ is epi and $U' \Vdash \phi(\alpha \circ f)$, then $U \Vdash \phi(\alpha)$

Proof:

If $U \Vdash \phi(\alpha)$, then $\alpha : U \rightarrow X$ factors through $\{x \mid \phi(x)\} \rightarrowtail X$. So clearly $\alpha \circ f : U' \rightarrow X$ also factors through $\{x \mid \phi(x)\} \rightarrowtail X$, so $U' \Vdash \phi(\alpha \circ f)$.

Let $f : U' \rightarrow U$ be epi. Consider two pullback squares:

$$\begin{array}{ccccc} Q & \twoheadrightarrow & P & \longrightarrow & \{x \mid \phi(x)\} \\ \downarrow n & & \downarrow m & & \downarrow \\ U' & \twoheadrightarrow_f & U & \xrightarrow{\alpha} & X \end{array}$$

If $U' \Vdash \phi(\alpha \circ f)$, then $\alpha \circ f$ factors through $\{x \mid \phi(x)\} \rightarrowtail X$ via some $U' \rightarrow \{x \mid \phi(x)\}$. The pair consisting of this map and the identity on U' factors through Q , it being a pullback, so $ns = \text{id}_{U'}$ for some $s : U' \rightarrow Q$, thus n is epic. Thus $f \circ n$ is epi, thus so is m , so m is an isomorphism, and we get that α factors through $\{x \mid \phi(x)\}$. ■

Throughout the rest of these notes, given a topological space X and open set U , I will use $\mathfrak{Y}(U)$ to denote the sheafification of the Yoneda embedding.

Example 4 (*Monotonicity and locality in $\text{Sh}(X)$*)

Let X be a topological space, $\mathcal{O}(X)$ the poset category of open subsets. Let $U \subseteq X$ be open. Given a formula $\phi(x) : \mathcal{X} \rightarrow \Omega$ and section $\alpha \in \mathcal{X}(U)$, let $U \Vdash \phi(\alpha)$ be shorthand for $\mathfrak{Y}(U) \Vdash \phi(\alpha)$, where we identify α with a map $\mathfrak{Y}(U) \rightarrow \mathcal{X}$. If $V \subseteq U$, there is a map $\mathfrak{Y}(i) :$

$\mathfrak{J}(V) \rightarrow \mathfrak{J}(U)$. Monotonicity says if $U \Vdash \phi(\alpha)$, then $V \Vdash \phi(\alpha|_V)$, which in an abuse of notation we will denote by $V \Vdash \phi(\alpha)$.

Corollary III.7.7 (Mac Lane and Moerdijk) says that given a family $\{U_i \hookrightarrow U\}_{i \in I}$, the induced map $\bigsqcup_{i \in I} \mathfrak{J}(U_i) \rightarrow \mathfrak{J}(U)$ is an epimorphism precisely when $\bigcup_{i \in I} U_i = U$. Take a section $\alpha \in \mathcal{X}(U)$. Then $\bigsqcup_{i \in I} \mathfrak{J}(U_i) \Vdash \phi(\alpha)$ implies $\mathfrak{J}(U) \Vdash \phi(\alpha)$. But the former condition means the map $\alpha : \bigsqcup_{i \in I} \mathfrak{J}(U_i) \rightarrow \mathcal{X}$ factors through the subobject classified by $\phi(x)$, which by the universal property of the coproduct is equivalent to the statement that each $\alpha : \mathfrak{J}(U_i) \rightarrow \mathcal{X}$ factors through this subobject. Thus,

$$U_i \Vdash \phi(\alpha) \text{ for all } i \in I \implies U \Vdash \phi(\alpha)$$

Together with monotonicity, we have:

$$\text{If } \{U_i\}_{i \in I} \text{ covers } U, \text{ then } U \Vdash \phi(\alpha) \text{ iff } U_i \Vdash \phi(\alpha) \text{ for each } i \in I$$

The following theorem gives a translation between the Mitchell-Bénabou language of a topos and more “familiar” language.

Theorem 1 (*Interpreting the Mitchell-Bénabou language*)

If $\alpha : U \rightarrow X$ is a generalized element of X while $\phi(x), \psi(x)$ are formulas with a free variable x of type X , then

1. $U \Vdash \phi(\alpha) \wedge \psi(\alpha)$ iff $U \Vdash \phi(\alpha)$ and $U \Vdash \psi(\alpha)$.
2. $U \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff there are arrows $p : V \rightarrow U, q : W \rightarrow U$ such that $p \sqcup q : V \sqcup W \twoheadrightarrow U$ is epi while both $V \Vdash \phi(\alpha p)$ and $W \Vdash \psi(\alpha q)$
3. $U \Vdash \phi(\alpha) \Rightarrow \psi(\alpha)$ iff for any $p : V \rightarrow U$ such that $V \Vdash \phi(\alpha p)$, also $V \Vdash \psi(\alpha p)$
4. $U \Vdash \neg \phi(\alpha)$ iff whenever $p : V \rightarrow U$ is such that $V \Vdash \phi(\alpha p)$, then $V \cong 0$.

Now let $\phi(x, y)$ be a formula with two free variables, the latter of type $\mathcal{Y}$. Then

5. $U \Vdash \exists y \phi(\alpha, y)$ iff there exist an epi $p : V \twoheadrightarrow U$ and generalized element $\beta : V \rightarrow Y$ such that $V \Vdash \phi(\alpha p, \beta)$
6. $U \Vdash \forall y \phi(\alpha, y)$ iff for every object V , for every arrow $p : V \rightarrow U$, and every generalized element $\beta : V \rightarrow Y$, one has $U \Vdash \phi(\alpha p, \beta)$

Proof:

The proof is quite long, and will be omitted in these notes. The basic idea is to use the correspondence between the internal Heyting algebra on Ω and the “external” Heyting algebra on sets of subobjects. For example, 1. is proven as follows: by definition of the meet operation of Ω , there is a pullback

$$\begin{array}{ccc} \{x \mid \phi(x) \wedge \psi(x)\} & \multimap & \{x \mid \phi(x)\} \\ \downarrow & & \downarrow \\ \{x \mid \psi(x)\} & \multimap & X \end{array}$$

Result 1. then follows from the universal property of pullbacks.

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Sheaf Semantics

The goal now is to extend Example 4 and interpret Theorem 1 for sheaves over a topological space. An example from Blechschmidt's dissertation is given.

Using the notation of Example 4, the expression " $U \Vdash \phi(\alpha)$ " means the dashed line exists:

$$\begin{array}{ccccc}
 & \{x \mid \phi(x)\} & \longrightarrow & 1 & \\
 & \downarrow & & \downarrow \text{true} & \\
 \mathfrak{J}(U) & \xrightarrow{\alpha} & \mathcal{X} & \xrightarrow{\phi(x)} & \Omega
 \end{array}$$

The final map $\mathfrak{J}(U) \rightarrow 1$ induces a logical morphism $\text{Sh}(X) \rightarrow \text{Sh}(X)_{/\mathfrak{J}(U)}$. Since 1 is terminal, this just results in taking the product with $\mathfrak{J}(U)$. Since everything is a $\mathfrak{J}(U)$ -morphism, it becomes clear that the dashed line exists in $\text{Sh}(X)$ iff it exists in $\text{Sh}(X)_{/\mathfrak{J}(U)}$. Note that $\mathfrak{J}(U) \times \mathfrak{J}(U) \rightarrow \mathfrak{J}(U)$ is isomorphic to $\text{id} : \mathfrak{J}(U) \rightarrow \mathfrak{J}(U)$, which is the terminal object. Thus under the equivalence $\text{Sh}(X)_{/\mathfrak{J}(U)} \simeq \text{Sh}(U)$, we have $U \Vdash \phi(\alpha)$ in $\text{Sh}(X)$ iff $1 \Vdash \phi(\alpha)|_U$ in $\text{Sh}(U)$ iff $\phi(\alpha)|_U = \text{true} \circ !_{\mathcal{X}|_U}$. In other words, $U \Vdash \phi(\alpha)$ iff $\phi(\alpha)|_U$ is universally valid in $\text{Sh}(U)$.

Example 5 ($U \Vdash \alpha = \beta$ iff $\alpha|_U = \beta|_U$)

Let $\alpha, \beta \in \mathcal{X}(X)$ be global sections of a sheaf. Then $\alpha = \beta$ is the composition

$$\mathfrak{J}(X) \xrightarrow{\langle \alpha, \beta \rangle} \mathcal{X} \times \mathcal{X} \xrightarrow{\delta_{\mathcal{X}}} \Omega$$

By the preceding paragraph, $U \Vdash \alpha = \beta$ is equivalent to the statement that $(\alpha = \beta)|_U$ is true, where $(\alpha = \beta)|_U$ will be the composition

$$\mathfrak{J}(U) \xrightarrow{\langle \alpha|_U, \beta|_U \rangle} \mathcal{X}|_U \times \mathcal{X}|_U \xrightarrow{\delta_{\mathcal{X}|_U}} \Omega$$

Since this is valid, it factors through the map $\text{true} : 1 \rightarrow \Omega$, so by the universal property of the pullback and the fact that true pulls back to $\Delta_{\mathcal{X}|_U}$ along $\delta_{\mathcal{X}|_U}$, we have that $\langle \alpha|_U, \beta|_U \rangle$ factors through Δ_U , so $\alpha|_U = \beta|_U$.

Given a global section $\alpha \in \mathcal{X}(X)$, the map $\phi(\alpha) = \alpha \circ \phi(x) : \mathfrak{J}(X) \rightarrow \Omega$ corresponds to an open set $U_{\phi(\alpha)}$ by Yoneda and the fact that $\Omega(W) = \{V \subseteq W \mid V \text{ open}\}$. We can interpret this as the largest open subset which forces $\phi(\alpha)$:

Proposition 2 (*Forcing and $U_{\phi(\alpha)}$*)

Given a global section $\alpha \in \mathcal{X}(X)$ for some sheaf $\mathcal{X}$, $U \Vdash \phi(\alpha)$ iff $U \subseteq U_{\phi(\alpha)}$.

Proof:

First we show $U_{\phi(\alpha)} \Vdash \phi(\alpha)$. Let $V \subseteq U_{\phi(\alpha)}$. We have the diagram

$$\begin{array}{ccc}
 \text{Hom}(X, X) & \xrightarrow{\phi(\alpha)_X} & \Omega(X) \\
 \downarrow & & \downarrow \\
 \text{Hom}(V, X) & \xrightarrow{\phi(\alpha)_V} & \Omega(V)
 \end{array}$$

where the right map is given by intersection with V and the left map is the obvious restriction. Note that $\text{Hom}(V, X)$ contains only a single point. Chasing the diagram,

$$(X \subseteq X) \mapsto U_{\phi(\alpha)} \mapsto U_{\phi(\alpha)} \cap V = V$$

Thus $\phi(\alpha)_V(V \subseteq X) = V$. We also have $\left(\text{true} \circ !_{\mathfrak{z}(U_{\phi(\alpha)})}\right)_V(V \subseteq X) = V$, thus we see

$$\phi(\alpha)|_{U_{\phi(\alpha)}} = \text{true} \circ !_{\mathfrak{z}(U_{\phi(\alpha)})}$$

Therefore $U_{\phi(\alpha)} \Vdash \phi(\alpha)$. By monotonicity, any $V \subseteq U_{\phi(\alpha)}$ forces $\phi(\alpha)$.

For the other direction, suppose $V \Vdash \phi(\alpha)$. Then $\phi(\alpha)|_V = \text{true} \circ !_{\mathfrak{z}(V)}$. To show $V \subseteq U_{\phi(\alpha)}$, by Yoneda it suffices to show that there is a map $\mathfrak{z}(V) \rightarrow \mathfrak{z}(U_{\phi(\alpha)})$. Thus it is sufficient to show the following is a pullback:

$$\begin{array}{ccc} \mathfrak{z}(U_{\phi(\alpha)}) & \longrightarrow & 1 \\ \downarrow & & \downarrow \text{true} \\ \mathfrak{z}(X) & \xrightarrow{\phi(\alpha)} & \Omega \end{array}$$

Since limits of sheaves are the pointwise limits, it suffices to check the square above is a pullback when applied to any open $U \subseteq X$. Suppose $U \subseteq U_{\phi(\alpha)}$. Then by the computation above, the bottom map sends $(U \subseteq X) \mapsto U$, which is also what true_U sends its element to. Since $\mathfrak{z}(X)(U) = \text{Hom}(U, X)$ and $1(U) = 1$ each have a single element which agree upon composition with $\phi(\alpha)_V$ and true_V , the pullback will be a single point, which $\mathfrak{z}(U_{\phi(\alpha)})(U) = \text{Hom}(U, U_{\phi(\alpha)})$ is. On the other hand, if $U \not\subseteq U_{\phi(\alpha)}$, then the bottom map sends $(U \subseteq X) \mapsto U_{\phi(\alpha)} \cap U$ while true_U sends its element to U . Since these do not agree, the pullback must be $\emptyset$, which is what $\mathfrak{z}(U_{\phi(\alpha)})(U)$ is in this case. ■

We can interpret Theorem 1 in the context of sheaves.

Theorem 2 (*Intpreting the Mitchell-Bénabou language in the category of sheaves*)

Let $\alpha \in \mathcal{X}(U)$ be a section of a sheaf $\mathcal{X}$, and let $\phi(x), \psi(x)$ be formulas from $\mathcal{X}$. Then:

1. $U \Vdash \phi(\alpha) \wedge \psi(\alpha)$ iff $U \Vdash \phi(\alpha)$ and $U \Vdash \psi(\alpha)$
2. $U \Vdash \phi(\alpha) \vee \psi(\alpha)$ iff there is a cover $\{U_i\}_{i \in I}$ of U such that for each i , $U_i \Vdash \phi(\alpha)$ or $U_i \Vdash \psi(\alpha)$
3. $U \Vdash \phi(\alpha) \Rightarrow \psi(\alpha)$ iff for any $V \subseteq U$ open, $V \Vdash \phi(\alpha)$ implies $V \Vdash \psi(\alpha)$
4. $U \Vdash \neg \phi(\alpha)$ iff whenever $V \subseteq U$ open is such that $V \Vdash \phi(\alpha)$, $V = \emptyset$

Now let $\phi(x, y)$ be a formula in two variables, the latter of type $\mathcal{Y}$. Then

5. $U \Vdash \exists y \phi(\alpha, y)$ iff there is a cover $\{U_i\}_{i \in I}$ of U and elements $\beta_i \in \mathcal{Y}(U_i)$ such that $U_i \Vdash \phi(\alpha, \beta_i)$ for all i
6. $U \Vdash \forall y \phi(\alpha, y)$ iff for every $V \subseteq U$ open and every $\beta \in \mathcal{Y}(V)$, $V \Vdash \phi(\alpha, \beta)$

Proof:

This follows directly from Theorem 1, locality, and the fact that every sheaf has an epimorphic cover by representable sheaves.

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Example 6 ($X \Vdash \neg\neg\phi(\alpha)$)

Consider a formula $\phi(x) : \mathcal{X} \rightarrow \Omega$ and section $\alpha \in \mathcal{X}(X)$. Using Theorem 2, we unpack what it means if $X \Vdash \neg\neg\phi(\alpha)$:

$$X \Vdash \neg\neg\phi(\alpha) \text{ iff } \neg\phi(\alpha) \text{ is valid only on } \emptyset.$$

The statement $\neg\phi(\alpha)$ is valid only on $\emptyset$ means $U \not\Vdash \neg\phi(\alpha)$ if $\emptyset \subsetneq U \subseteq X$. This in turn means there exists some nonempty $V \subseteq U$ on which $\phi(\alpha)$ holds. Thus

$$X \Vdash \neg\neg\phi(\alpha) \text{ iff for any open } U \subseteq X, \text{ there exists some open } V \subseteq U \text{ such that } V \Vdash \phi(\alpha).$$

By locality and Proposition 2, we see that

$$X \Vdash \neg\neg\phi(\alpha) \text{ iff } U_{\phi(\alpha)} \text{ is dense in } X.$$

We will conclude with a sketch of the following result from Blechschmidt's dissertation *Using the internal language of toposes in algebraic geometry*:

Lemma (*Finite locally free on dense open*)

Let X be a reduced scheme. Let $\mathcal{F}$ be a $\mathcal{O}_X$ -module of finite type. Then $\mathcal{F}$ is locally finite free on a dense open subset.

Proof:

A reduced scheme means $\mathcal{O}_X$ is reduced as an internal ring i.e.

$$X \Vdash \forall s : \mathcal{O}_X. \left(\bigvee_{n \geq 0} s^n = 0 \right) \Rightarrow s = 0$$

An $\mathcal{O}_X$ -module $\mathcal{F}$ is of finite locally free iff $\mathcal{F}$ is internally a free $\mathcal{O}_X$ -module, i.e.

$$X \Vdash \bigvee_{n \geq 0} \exists x_1, \dots, x_n : \mathcal{F}. \forall x : \mathcal{F}. \exists ! a_1, \dots, a_n : \mathcal{O}_X. x = \sum_i a_i x_i$$

An $\mathcal{O}_X$ -module $\mathcal{F}$ is of finite type iff $\mathcal{F}$ is internally a finitely-generated $\mathcal{O}_X$ -module, i.e.

$$X \Vdash \bigvee_{n \geq 0} \exists x_1, \dots, x_n : \mathcal{F}. \forall x : \mathcal{F}. \exists a_1, \dots, a_n : \mathcal{O}_X. x = \sum_i a_i x_i$$

Because X is reduced, $\mathcal{O}_X$ is a field from the internal view, i.e.

$$X \Vdash \forall s : \mathcal{O}_X. \neg("s \text{ invertible}") \Rightarrow s = 0$$

The following statement has a constructive proof:

Let R be a local ring satisfying the property from the previous line. Let M be a f.g. R -module.

Then M is not not finite free.

Translating the statement using the semantics in Theorem 2 yields the desired result.

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