

Profinite Negligible Cohomology and the Inverse Absolute Galois Problem

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Abstract

Let F be a field, K/F a field extension with $\Gamma_K := \text{Gal}(K^{\text{sep}}/K)$, G a profinite group, $j : \Gamma_K \rightarrow G$ a continuous group homomorphism, and M an abelian group. Then there is an induced map $j^* : H^n(G; M) \rightarrow H^n(K; M) := H^n(\Gamma_K; M)$. We say an element $u \in H^n(G; M)$ is negligible over F if $j^*(u) = 0$ for all such j and field extensions K/F . The space of negligible elements forms a subgroup $H^n(G; M)_{\text{neg}} \subseteq H^n(G; M)$. In this paper, we use negligible cohomology to prove that certain groups cannot be absolute Galois groups. We first develop the basic theory of profinite negligible cohomology. We then show how the theory of negligible cohomology of finite groups can be used to prove the classic Artin-Schreier theorem of Galois theory. Finally, we compute the negligible cohomology of certain Demushkin groups over a specific field of characteristic zero, and show that a Demushkin pro- p -group cannot be the absolute Galois group of any field of characteristic p .

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1. Introduction

Substantial literature was written regarding the inverse Galois problem and its variants (see [1] for a survey of known results). In this note we are concerned with the problem of determining which groups can be absolute Galois groups. In particular, we consider a base field F , and aim to determine which groups can be absolute Galois groups of fields over F . Let Γ_F denote the absolute Galois group of F .

A classical result imposing a restriction on the groups which can be absolute Galois groups is the Artin-Schreier Theorem:

Theorem 1.1: Let F be a field and $\overline{F}$ be its algebraic closure. If

$$1 < [\overline{F} : F] < \infty, \tag{1}$$

then F has characteristic 0 and $\overline{F} = F(\sqrt{-1})$.

This completely classifies the case for finite groups, with only the trivial group and $\mathbb{Z}/2\mathbb{Z}$ appearing as absolute Galois groups. For the infinite case, an immediate restriction is that the group is profinite. Unlike the finite case, a multitude of profinite groups can be absolute Galois groups. This is a consequence of a theorem by Neukirch and Uchida [2], [3], as there are at countably many number fields which distinct up to isomorphism:

Theorem 1.2: Two number fields are isomorphic if and only if their absolute Galois groups are isomorphic as topological groups.

Then, in view of the fact that there are many possible profinite Galois groups, we now turn our attention to restricting which groups are possible. There are important results and conjectures regulating the possible absolute Galois groups.

First, Voevodsky's [4] norm-residue isomorphism theorem states that the map:

$$K_d^M(F)/\ell K_d^M(F) \longrightarrow H^d(F; \mu_\ell^{\otimes d}) \quad (2)$$

must be an isomorphism in all degrees, where $K_d^M(F)$ is the d -th Milnor K -theory of the field F .

Secondly, the generalized Bogomolov conjecture, as proposed by Positselski in his PhD thesis [5], states the following:

Conjecture 1.1 (Bogomolov–Positselski): If F is a field, ℓ is a prime number and $K = F[\sqrt[\ell]{F}]$ is the field obtained from F by adjoining to F all ℓ -th power roots of elements in it. Then, the Sylow pro- ℓ subgroup of Γ_K is a free pro- ℓ group.

For the original conjecture by Bogomolov, see [6] or [5]. At the time of publication of [5], the norm-residue isomorphism was still conjectural, and Positselski showed that the Bogomolov–Positselski conjecture implies both the norm-residue isomorphism and Bogomolov's original conjecture.

As a side result, Positselski [7] in an extension of his thesis [5] also shows that any absolute Galois group G admits a finite filtration $G \supset G_0 \supset G_1 \supset \cdots \supset G_N \supset G_{N+1} = \{1\}$ by closed normal subgroups such that G/G_0 is either trivial or $\mathbb{Z}/2\mathbb{Z}$ and for $n \geq 0$, G_n/G_{n+1} is a closed subgroup of a free profinite group, also

called a *projective* profinite group, which are the profinite groups of cohomological dimension at most 1.

Furthermore, Efrat [8] has conjectured the full class of absolute Galois groups which are finitely generated pro- ℓ groups to be the so-called “elementary type” groups. For this we consider pro- ℓ groups G equipped with an orientation, i.e., a map $\theta : G \rightarrow \mathbb{Z}_\ell^\times$. The set of elementary type pro- ℓ groups is the smallest set containing $\mathbb{Z}_\ell$ with all its orientations, all Demushkin groups with their natural orientation, and which is closed under free and fiber products in the category of oriented pro- ℓ groups.

Indeed, in [9], Quadrelli and Weigel show that every oriented elementary type pro- ℓ group satisfies the hypotheses of the Bogomolov–Positselski theorem (which can be rewritten in terms of the natural orientation of the maximal pro- ℓ quotient of Γ_K).

To impose further restrictions on the groups which can be absolute Galois groups, we use negligible cohomology groups, in a slight variation to Jean-Pierre Serre’s original definition in [10, §26]. This variation is given in Definition 1.1.

For a profinite group G and discrete G -module M , $H^d(G; M)$ denotes the continuous cohomology of G with M coefficients. When G is finite, this is just the group cohomology. When F is a field, M a Γ_F -module, $H^d(F; M) := H^d(\Gamma_F; M)$ is the Galois cohomology of Γ_F . By a trivial G -module, we mean a discrete G -module with trivial G -action.

Definition 1.1: Let G be a profinite group, M a discrete G -module, and F a field. We say that $u \in H^d(G; M)$ is negligible, or a negligible class, over F if for all extensions K/F and continuous group homomorphisms $\phi : \Gamma_K \rightarrow G$,

$$u \in \ker(\phi^* : H^d(G; M) \rightarrow H^d(K; M)), \quad (3)$$

where Γ_K is the absolute Galois group of K . We denote the subgroup of negligible classes of F in degree d by $H^d(G; M)_{\text{neg}, F}$ and call this the “negligible cohomology of G with M coefficients in degree d over F .” When there is no ambiguity, we omit the base field. We say that $H^d(G; M)$ is entirely negligible over F if $H^d(G; M)_{\text{neg}, F} = H^d(G; M)$. When $M = \mathbb{Z}/p\mathbb{Z}$ and has trivial G -action, we use the shorthand $H^d(G; p) := H^d(G; \mathbb{Z}/p\mathbb{Z})$.

Following Serre’s initial work in [10, §26] and using the definition of negligible cohomology given above, Saltman studied and computed the degree three negligible cohomology of a finite group G with $\mathbb{Q}/\mathbb{Z}$ coefficients over $\mathbb{C}$ [11]. Later on, Peyre constructed nontrivial degree three negligible classes over fields not of characteristic two with fourth roots of unity for groups appearing as the central extension of an $\mathbb{F}_2$ -vector space by another [12]. More recently, Gherman and Merkurjev studied negligible cohomology of finite groups over a more general class of fields, providing a complete description of degree two negligible classes over fields with sufficiently many roots of unity in [13], and computing the Krull dimension of the quotient of the cohomology ring $H^*(G; \mathbb{Z}/p\mathbb{Z}) := \bigoplus_{d \in \mathbb{N}} H^d(G; p)$ by the ideal of negligible classes over fields with enough roots of unity for most finite groups G .

The following proposition – despite its straightforward proof – is the crux of our method.

Proposition 1.1: Let G be a profinite group. If G is the absolute Galois group of a field K , then the negligible cohomology of G with any coefficients over a subfield F of K is trivial.

Proof: Let Γ_K denote the absolute Galois group of K . If $\phi : \Gamma_K \rightarrow G$ is a continuous isomorphism, then for each $d \geq 0$ the map $\phi^* : H^d(G; M) \rightarrow H^d(K; M)$ is an isomorphism in Grp and Top and so has trivial kernel for any Γ_K -module M . Thus there exists an extension K/F and continuous map $\phi : \Gamma_K \rightarrow G$ such that $\ker \phi^* = 0$.

Since $\ker \phi^*$ is included in the intersection of kernels defining negligible cohomology of G over K , $H^d(G; M)_{\text{neg}, F} = 0$. ■

Thus to show a profinite group G is not an absolute Galois group, it suffices to show it has a nontrivial negligible cohomology class. Indeed, by the fundamental theorem of Galois theory, we need even less:

Corollary: Let G be the absolute Galois group of a field K . Then any closed subgroup of G has trivial negligible cohomology over any subfield of K .

Proof: Note that any closed subgroup of an absolute Galois group is also an absolute Galois group. ■

The structure of the remainder of this paper is the following: in section 2, we develop the theory of profinite negligible cohomology, including generalizing results in [13], [14] to profinite groups. In section 3, we provide a new proof of the classic Artin-Schreier theorem using tools from negligible cohomology (for this section, only the theory for finite groups is required). Finally, in section 4 we calculate the negligible classes of Demushkin pro- p -groups over fields of characteristic p and a certain class of Demushkin groups over a specific field of characteristic zero.

2. Profinite Negligible Cohomology Theory

We first extend [13, Proposition 2.3 (1),(2),(4)] to the profinite case. Noting that any profinite group can be realized as the cone of a cofiltered system of finite groups, we then prove a relationship between the negligible classes of these finite groups and the negligible classes of the profinite group. Finally, we state a few cohomological conditions which imply a group cannot be an absolute Galois group.

Proposition 2.1: Let G be a profinite group, M a trivial G -module and K/F be a field extension. Then $H^d(G, M)_{\text{neg}, F} \subseteq H^d(G, M)_{\text{neg}, K}$.

Proof: Note that any extension of K is an extension of F , so for u to be negligible over F , it has to vanish under all maps from extensions of K . Therefore, u has to be negligible over K . ■

Proposition 2.2: Let $\phi : G \rightarrow G'$ be a continuous homomorphism between profinite groups, M a G' -module, and F a field. Let

$$\phi^* : H^d(G'; M) \rightarrow H^d(G; M) \quad (4)$$

denote the induced map on cohomology. Then $u \in H^d(G'; M)_{\text{neg}, F}$ implies $\phi^*(u) \in H^d(G; M)_{\text{neg}, F}$.

Proof: Let K/F be any field extension, and let $\psi : \Gamma_K \rightarrow G$ be a continuous group homomorphism. Take $u \in H^d(G'; M)_{\text{neg}, F}$. Then $\phi \circ \psi : \Gamma_K \rightarrow G'$ is a continuous group homomorphism, so $(\phi \circ \psi)^*(u) = \psi^* \circ \phi^*(u) = 0$ by definition of negligible classes. Thus $\phi^*(u) \in \ker \psi^*$. Since the choice of extension and continuous group map were arbitrary, $\phi^*(u) \in H^d(G; M)_{\text{neg}, F}$. ■

Proposition 2.3: Let G be a profinite group, $\varphi : M \rightarrow N$ a map of G -modules, and F a field. Let $\varphi_* : H^d(G; M) \rightarrow H^d(G; N)$ be the induced map. Then if $u \in H^d(G; M)_{\text{neg}, F}$, we have $\varphi_*(u) \in H^d(G; N)_{\text{neg}, F}$.

Proof: Let K/F be any field extension and $\psi : \Gamma_K \rightarrow G$ any continuous group map. It is a well-known fact that the following commutes:

$$\begin{array}{ccc} H^d(G; M) & \xrightarrow{\varphi_*^{(G)}} & H^d(G; N) \\ \downarrow \psi_{(M)}^* & & \downarrow \psi_{(N)}^* \\ H^d(K; M) & \xrightarrow{\varphi_*^{(\Gamma_K)}} & H^d(K; N) \end{array}$$

Where here we use the upper and lower indicators of the induced maps for clarity of notation. Take $u \in H^d(G; M)$. Then:

$$\psi_{(M)}^*(u) = 0 \implies \varphi_{*}^{(\Gamma_K)} \psi_{(M)}^*(u) = \psi_{(N)}^* \varphi_{*}^{(G)}(u) = 0 \quad (5)$$

Thus $\varphi_{*}^{(G)}(u) = \varphi_{*}(u) \in \ker(\psi_{(N)}^* = \psi^*)$. Since K/F and ψ are arbitrary, it follows that $\varphi_{*}(u) \in H^d(G; N)_{\text{neg}, F}$. $\blacksquare$

We now extend [13, Corollary 2.2] to the profinite case. Since the algebraic-geometric tools used in [13, Proposition 2.1] do not immediately carry over to the profinite case, we have to use a slightly different method.

Proposition 2.4: Let G be a profinite group, F a field, and M a discrete G -module. Then $H^d(G; M)_{\text{neg}, F} = 0$ for $d \leq 1$.

Proof: Note that all profinite groups are quotients of free profinite groups by closed normal subgroups. Since free profinite groups are projective, by [15, Theorem 23.1.1] any free group is an absolute Galois group of some field K which is an extension of F . Hence we have a continuous map $\phi : \Gamma_K \twoheadrightarrow G$ which must have a closed kernel, so by the Hochschild-Serre spectral sequence for Galois cohomology, there is an exact sequence

$$0 \rightarrow H^1(\Gamma_K / \ker \phi; M) \cong H^1(G; M) \xrightarrow{\phi^*} H^1(K; M) \quad (6)$$

Thus there exists an extension K/F and continuous homomorphism $\phi : \Gamma_K \rightarrow M$ such that the induced map on cohomology has trivial kernel. By Definition 1.1, it follows that $H^1(G; M)_{\text{neg}, F} = 0$. When $d = 0$, this is just a consequence of left-exactness of the invariant subgroup functor $\blacksquare$

Now we give a general result for negligible classes over fields of characteristic p which we will use in section 4.

Lemma 2.1: Let K be a field of characteristic p . Then $H^d(K; p) = 0$ for all $d > 1$.

Proof: Given that K has characteristic p , we have a short exact sequence of additive groups

$$0 \rightarrow \mathbb{F}_p \rightarrow K^{\text{sep}} \rightarrow K^{\text{sep}} \rightarrow 0 \quad (7)$$

Where the first map sends $1 \mapsto 1$ and the latter map sends $x \mapsto x^p - x$. Viewing $\mathbb{F}_p$ as a subfield of K^{sep} , it has a canonical Γ_K action. Since $\mathbb{F}_p \subseteq K$, this action is trivial. Thus $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ as (trivial) Γ_K -modules. From the long exact sequence in Galois cohomology

$$\dots \rightarrow H^1(K; K^{\text{sep}}) \rightarrow H^2(K; p) \rightarrow H^2(K; K^{\text{sep}}) \rightarrow \quad (8a)$$

$$\dots \rightarrow H^{d-1}(K; K^{\text{sep}}) \rightarrow H^d(K; p) \rightarrow H^d(K; K^{\text{sep}}) \rightarrow \dots \quad (8b)$$

By Additive Hilbert 90 [16, Theorem 6.11.16], $H^d(K; K^{\text{sep}}) = 0$ for all $d > 0$, so by exactness $H^d(K; p) = 0$ for all $d > 1$. ■

Corollary: Let G be a profinite group and F a field of characteristic p . Then $H^d(G; p)$ is entirely negligible over F for any $d > 1$.

Proof: Let K/F be any field extension and $\phi : \Gamma_K \rightarrow G$ be any continuous group homomorphism. Then the kernel of $\phi^* : H^d(G; p) \rightarrow H^d(K; p)$ must be everything since $H^d(K; p) = 0$ by Lemma 2.1. By arbitrariness of K/F and ϕ , the intersection of all kernels is $H^d(G; p)$, hence the result. ■

We now show how the negligible cohomology of the continuous finite quotients of a profinite group contributes to the overall cohomology.

Proposition 2.5: Let $G := \varprojlim G_i$ for some collection of groups G_i with legs $\pi_i : G \rightarrow G_i$. Fix a field F . Then we have:

$$\bigcup_{i \in I} \pi_i^* \left(H^d(G_i; M)_{\text{neg}, F} \right) \subseteq H^d(G; M)_{\text{neg}, F} \quad (9)$$

Proof: Consider an element $x \in \bigcup_{i \in I} \pi_i^* \left(H^d(G_i; M)_{\text{neg}} \right)$. Then $x = \pi_i^*(y)$ for some $i \in I$ and $y \in H^d(G_i; M)_{\text{neg}}$, and the result follows immediately from Proposition 2.1. ■

It is currently unknown if equality holds in Proposition 2.5, although we expect that it will not hold true in general, as this would require strong lifting properties of continuous group homomorphisms. We do have the following basic result, however.

Remark 2.1: In order for equality in Proposition 2.5 to hold, it is sufficient that $H^d(G_i; M)_{\text{neg}, F} = H^d(G_i; M)$ for each $i \in I$.

Proof: We use the same notation as in the proof of Proposition 2.5. Using basic properties of Galois cohomology:

$$H^d(G; M) \cong \varinjlim H^d(G_i; M) \quad (10)$$

Where the legs of the direct limit are the induced maps $\pi_i^* : H^d(G_i; M) \rightarrow H^d(G; M)$. By [16, Lemma 2.6.14], we have:

$$H^d(G; M) = \bigcup_{i \in I} \pi_i^*(H^d(G_i; M)) \quad (11)$$

By assumption:

$$H^d(G_i; M)_{\text{neg}, F} = H^d(G_i; M) \quad (12)$$

Therefore by Proposition 2.5 we have:

$$\bigcup_{i \in I} \pi_i^*(H^d(G_i; M)_{\text{neg}, F}) = \bigcup_{i \in I} \pi_i^*(H^d(G_i; M)) \quad (13a)$$

$$= H^d(G; M) \subseteq H^d(G; M)_{\text{neg}, F} \quad (13b)$$

Therefore the inclusion of Proposition 2.5 is an equality. ■

We now present a cohomological condition which forces a profinite group to not be an absolute Galois group over a given field.

Proposition 2.6: Let k be a field. Let G be a profinite group, $H \triangleleft G$ be a closed normal subgroup, and M a trivial G -module. If there exists some $D \in \mathbb{N}$ such that $H^d(H; M) = 0$ for all $1 \leq d < D$ and $H^D(G/H; M)$ contains a nontrivial negligible class over F , then G cannot be an absolute Galois group of any extension K/F .

Proof:

Let $\phi : G \twoheadrightarrow G/H$ be the canonical quotient map. By assumption, H is closed, thus this map is continuous. Also by assumption, $H^d(H; M) = 0$ for all $1 \leq d < D$, so the Hochschild-Serre spectral sequence implies

$$H^{D-1}(H; M) = 0 \rightarrow H^D(G/H; M) \xrightarrow{\phi^*} H^D(G; M) \quad (14)$$

Since $H^D(G/H; M)$ has a nontrivial class (call it u) and by Proposition 2.2, $\phi^*(u) \in H^D(G; M)_{\text{neg}, F}$. Furthermore, ϕ^* is injective, thus $\phi^*(u) \neq 0$,

so $H^D(G; M)_{\text{neg}, K} \neq 0$, thus by Proposition 1.1 G cannot be the absolute Galois group of any extension K/F . ■

Propositions 2.5 and 2.6 allow one to compute negligible classes of a profinite group by considering finite quotients by closed subgroups, for which many results are known, see [13] and [14]. We have not yet found any groups for which these propositions yield novel results, but we have been able to use the condition to show that certain products of profinite groups and finite groups cannot be absolute Galois groups. However, this result is an immediate corollary of the classic Artin-Schreier theorem.

3. Finite Absolute Galois Groups

For this section, we deal only with finite groups. Hence, the tools from [13],[14] apply. Since the algebraic closure of a field contains its separable closure, the following implies the Artin-Schreier theorem:

Theorem 3.1: Let F be a field and F^{sep} be its separable closure. Assume $1 < [F^{\text{sep}} : F] < \infty$. Then, F has characteristic 0 and $F^{\text{sep}} = F(\sqrt{-1})$.

Proof: Let $G := \text{Gal}(F^{\text{sep}}/F)$. First, we show that if G is not a 2-group, then G is not an absolute Galois group.

Let p be an odd prime divisor of G ; such a p exists by the assumption that G is not a 2-group. By [14, Theorem 2.2], there exists $N \in \mathbb{N}$ s.t.

$$H^d(G; p)_{\text{neg}, \mathbb{Q}} = H^d(G; p) \tag{15}$$

for all $d > N$. By [17], there exists an $M \in \mathbb{N}$ with $M > d$ such that $H^M(G; p) \neq 0$. Thus G has a nontrivial negligible class over $\mathbb{Q}$, and so by

[10, Proposition 26.1], G also has nontrivial negligible classes over $\mathbb{F}_q$ for each prime q . It follows from Proposition 1.1 that G cannot be an absolute Galois group.

Now assume G is a 2-group. Assume for contradiction G has order at least 4. Since it is a 2-group, it has a subgroup of order 4, which must be the absolute Galois group of a finite extension of F by the fundamental theorem of Galois theory. Hence, assume without loss of generality that G has order 4.

If G is elementary abelian, by [14, Theorems 3.2 and 4.3], we have that the Krull dimension of the negligible quotient of $H^*(G, \mathbb{Z}/2\mathbb{Z})$ is at most 1. Further, by [17], $H^*(G, \mathbb{Z}/2\mathbb{Z})$ itself has Krull dimension at least 2, so there must be negligible classes in arbitrarily large degree, contradicting Proposition 1.1.

If G is cyclic of order 4, then it is known that G has negligible classes with $\mathbb{Z}/2\mathbb{Z}$ coefficients over $\mathbb{Q}$. By [10, Proposition 26.1], this implies that it has negligible classes over any base field. Hence, G is not an absolute Galois group.

This leaves us with the case where $G \cong \mathbb{Z}/2\mathbb{Z}$. If the characteristic of F is 2, then by the Corollary to Lemma 2.1, $H^d(G; \mathbb{Z}/2\mathbb{Z})$ is entirely negligible over $\mathbb{F}_2$ and by [17] it is nontrivial for infinitely many d . Hence G has negligible classes, so is not an absolute Galois group by Proposition 1.1.

Hence, $\text{char}(F) \neq 2$. Note that for all $d \geq 0$:

$$H^d(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \tag{16}$$

is nontrivial. If F has 4-th roots of unity, then by [14, Theorem 3.2], $H^d(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$ is entirely negligible over F for some d . Therefore, G cannot be an absolute Galois group.

Otherwise, $\sqrt{-1}$ is separable, and $F(\sqrt{-1})/F \cong F^{\text{sep}}/F$. Let $p = \text{char } F$, and assume $p > 0$. Note that $\mathbb{F}_p^{\text{sep}}$ is contained in F^{sep} . Further, since F has no 4-th roots of unity, it does not contain $\mathbb{F}_p^{\text{sep}}$. Hence, restricting the nontrivial automorphism in G to $\mathbb{F}_p^{\text{sep}}$ yields a nontrivial element of finite order in $\text{Gal}(\mathbb{F}_p^{\text{sep}}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$, a contradiction. ■

In particular, the proof of this theorem shows that a finite group is an absolute Galois group of an extension of a base field F if and only if for every subgroup H of G , $H^d(H; M)_{\text{neg}, F} = 0$ for all $d \geq 1$ and discrete H -modules M .

4. Negligible Cohomology for Demushkin Groups

In this section, we perform some elementary calculations of the negligible cohomology of Demushkin groups. We first define them.

Definition 4.1: A Demushkin group Γ is a pro- p -group such that $H^1(\Gamma; p)$ has finite rank, $H^2(\Gamma; p)$ has rank one, and

$$\cup : H^1(\Gamma; p) \otimes_{\mathbb{Z}} H^1(\Gamma; p) \rightarrow H^2(\Gamma; p) \quad (17)$$

is nondegenerate.

For example, it is shown in [18] that the Galois group of a maximal p -extension of a local number field containing all p -th roots of unity is a Demushkin group. When G is a Demushkin group of this form for some field K , we can calculate its negligible cohomology over K .

Proposition 4.1: Let Γ be a Demushkin pro- p -group appearing as the Galois group described in Definition 4.1 for a local number field K . Then $H^d(\Gamma; p)_{\text{neg}, K} = 0$ for all $d \geq 1$.

Proof: Let Γ_K be the separable closure of K . By Galois correspondence, Γ appears as a quotient of Γ_K by some closed normal subgroup N . Let $\phi : \Gamma_K \twoheadrightarrow \Gamma$ be this continuous homomorphism. Furthermore, N has no p -torsion since Γ is the Galois group of the maximal p -extension of K . Thus $H^d(N; p) = 0$ for all $d \geq 1$. By the Hochschild-Serre spectral sequence for profinite groups, since our action is trivial, we have an exact sequence

$$0 \rightarrow H^1(\Gamma; p) \xrightarrow{\phi^*} H^1(K; p) \rightarrow H^1(N; p)^\Gamma \rightarrow H^2(\Gamma; p) \xrightarrow{\phi^*} H^2(K; p) \quad (18)$$

Since $H^1(N; p) = 0$, the middle term vanishes, thus ϕ^* is injective on degree two, and the spectral sequence can be restarted at $d = 2$. Continuing this process, we have for each $d > 1$ the exact sequence:

$$H^{d-1}(N; p)^\Gamma = 0 \rightarrow H^d(\Gamma; p) \xrightarrow{\phi^*} H^d(K; p) \quad (19)$$

It follows from Eq. (18) and Eq. (19) that $\phi^* : H^d(\Gamma; p) \rightarrow H^d(K; p)$ is injective for all $d \geq 1$, so $\ker \phi^* = 0$ in each degree. Since negligible classes are the intersection of kernels of extensions, we have $H^d(\Gamma; p)_{\text{neg}, K} = 0$ for all $d \geq 1$. ■

Using the tools developed in Section 2, we can also show that a Demushkin pro- p -group cannot be the absolute Galois group of a field of characteristic p .

Proposition 4.2: Let Γ be a Demushkin pro- p -group. Then Γ cannot be the absolute Galois group of any field of characteristic p .

Proof: Let F be a characteristic p field. We have from the Corollary of Lemma 2.3 that $H^d(\Gamma; p)_{\text{neg}, F} = H^d(\Gamma; p)$ for all $d > 1$. By definition, $H^2(\Gamma; p) \neq 0$, thus $H^2(\Gamma; p)$ has a nontrivial negligible class over F . The result follows from Proposition 1.1. ■

By Remark 2.1 and the corollary of Lemma 2.1, these Demushkin groups are an example of a class of groups with nontrivial cohomology for which equality in Proposition 2.5 holds.

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