

Yoneda's Lemma for Internal Higher Categories

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Motivation

- “=” is bad and goes against the spirit of category theory
- Categories let us look at things “up to isomorphism”
- Weaken from $f \circ g = \text{id}$ to $f \circ g \cong \text{id}$ (isomorphic as maps)
- Continuing with this reasoning, we run into problems:
isomorphisms should be compatible in some sense
- ∞ -categories formalize handling this
- And we remove pesky equalities: “=” is the same as “ $\simeq$ ”

Simplicial Category

Definition (Simplicial Category Δ)

Objects are ordered sets $\{0, \dots, n\}$ for $n \in \mathbb{N}$. Morphisms are order-preserving functions.

- Morphisms can be decomposed into face and degeneracy maps
- Degeneracy maps are order-preserving functions $s^i : [n+1] \rightarrow [n]$ which repeat i
- Face maps are order-preserving functions $d^i : [n] \rightarrow [n+1]$ which skip i

Simplicial Sets

Definition (Simplicial Set)

The category of simplicial sets is $\mathbf{sSet} := \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$.

- We have Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathbf{sSet}$
- Let $\Delta^n = \mathrm{Hom}(-, [n])$. For F a simplicial set, define $F_n := F([n]) \cong \mathrm{Hom}(\Delta^n, F)$ (last isomorphism is Yoneda)
- Like other presheaf categories, $F = \mathrm{colim}_{\Delta^n \rightarrow F} \mathrm{Hom}(-, \Delta^n)$

Simplicial Sets (cont.)

- A simplicial set F has a geometric realization $\operatorname{colim}_{\Delta^n \rightarrow F} \Delta^n_{\text{top}}$, where $\Delta^n_{\text{top}} \subseteq \mathbb{R}^n$ is the standard n -simplex
- Face and degeneracy induce expected face and degeneracy maps of Δ^n_{top}
- A topological space X has an associated simplicial set $\operatorname{Sing}_\bullet(X)$ given by $\operatorname{Sing}_n(X) := \{\Delta^n_{\text{top}} \rightarrow X \text{ continuous}\}$
- These are adjoint: $\operatorname{Hom}_{\mathbf{sSet}}(F, \operatorname{Sing}_\bullet(X)) \cong \operatorname{Hom}_{\mathbf{Top}}(|F|, X)$

∞ -Categories

- The horn Λ_k^n is the simplicial set Δ^n with the interior and face opposite the k -th 0-simplex removed

Definition (∞ -Category)

An ∞ -category (also called higher category) is a simplicial set $\mathcal{C}$ satisfying the “horn filling condition” for $0 < n < k$:

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

∞ -Categories (cont.)

- Lets us compose morphisms in a homotopy coherent way
- Horn filling for Λ_1^2 is like composition
- Composition is not unique: main distinction from ordinary category theory
- Kan complexes are similar: have horn filling for all $0 \leq k \leq n$
- For any two $x, y : \Delta^0 \rightarrow \mathcal{C}$, there is a space $\mathrm{map}_{\mathcal{C}}(x, y)$ of maps whose objects are maps $f : \Delta^1 \rightarrow \mathcal{C}$ with $d^1 f = x, d^0 f = y$
- All 1-categorical constructions work (Kan extensions, (co)limits, adjoints, etc.)

Examples of ∞ -Categories

Definition (Nerve of a Category)

The nerve of a category C is the simplicial set $N(C)$ whose n -simplices are given by $\mathrm{Fun}_{\mathrm{Cat}}([n], C)$.

- n -simplices are sequences of n composable morphisms.
- Composition in C lets us fill inner horns, so $N(C)$ is an ∞ -category
- $|\Lambda_k^n|$ is a strong deformation retract of Δ_{top}^n , which shows $\mathrm{Sing}_\bullet(X)$ satisfies all horn filling conditions, so spaces are Kan complexes

The ∞ -Category of Spaces

- Want ∞ -category version of Set (0-groupoids)

Definition (∞ -groupoid)

An ∞ -groupoid is an ∞ -category $\mathcal{C}$ in which every map $\Delta^1 \rightarrow \mathcal{C}$ is an equivalence.

Theorem

∞ -groupoids are Kan complexes (and spaces)

- We can define ∞ -category $\mathcal{S}$ of ∞ -groupoids

∞ -Topoi

- Want ∞ -categories which behave “like” $\mathcal{S}$
- These are “parameterized” spaces or sheaves of spaces

Definition (∞ -topos)

Let $\mathcal{G}$ be a small ∞ -category. An ∞ -topos is an ∞ -category $\mathcal{B}$ which arises as a left exact accessible localization: there is an embedding $\mathcal{B} \hookrightarrow \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S})$ which admits a left adjoint $L : \mathrm{Fun}(\mathcal{G}^{\mathrm{op}}, \mathcal{S}) \rightarrow \mathcal{B}$ which commutes with finite limits and $Li \simeq \mathrm{id}_{\mathcal{B}}$.

Important Adjunction

There is an adjunction

$$\mathrm{const}_{\mathcal{B}} \dashv \Gamma : \mathcal{B} \rightarrow \mathcal{S}$$

- $\Gamma = \mathrm{map}_{\mathcal{B}}(1, -)$ sends an object $A \in \mathcal{B}$ to the space of all maps from the final object $1 \in \mathcal{B}$ to A
- $\mathrm{const}_{\mathcal{B}}$ sends a space $X \in \mathcal{S}$ to the localization of the presheaf sending every object to X

Motivation

- Ordinary higher category theory is done in the ∞ -topos $\mathcal{S}$
- Sometimes we want to work in other ∞ -topoi, such as topos of spaces with group action, or topos of parameterized spaces, or topos sheaves of spaces
- Many constructions look the same “inside” a topos, and many proofs can be done “inside” a topos
- For example, internal categories in $\mathcal{S}^G$ are ∞ -categories with a G -action

Simplicial $\mathcal{B}$ -objects

Fix an ∞ -topos $\mathcal{B}$.

Definition (Simplicial Object)

The ∞ -category of simplicial objects is $\mathcal{B}_\Delta := \text{Fun}(N(\Delta)^{\text{op}}, \mathcal{B})$

- Let F_n be the evaluation of F at the object $[n] \in \Delta^{\text{op}}$
- Then F_0 is a $\mathcal{B}$ -object of objects
- F_1 is a $\mathcal{B}$ -object of morphisms
- $F_n, n \geq 2$ is a $\mathcal{B}$ -object of higher coherence data
- An internal category of $\mathcal{B}$ should be a simplicial $\mathcal{B}$ -object with composition (more on this later)

Important Functors

- $[-, -] = \underline{\text{Fun}}_{\mathcal{B}}(-, -) : \mathcal{B}_{\Delta}^{\text{op}} \times \mathcal{B}_{\Delta} \rightarrow \mathcal{B}_{\Delta}$ is internal hom
- $- \otimes - : \mathcal{S}_{\Delta} \times \mathcal{B}_{\Delta} \rightarrow \mathcal{B}_{\Delta}$, given by $(- \times -) \circ (\text{const} \times \text{id})$
- $(-)^{(-)} : \mathcal{S}_{\Delta}^{\text{op}} \times \mathcal{B}_{\Delta} \rightarrow \mathcal{B}_{\Delta}$, given by $[-, -] \circ (\text{const} \times \text{id})$
- $\text{Fun}_{\mathcal{B}}(-, -) : \mathcal{B}_{\Delta}^{\text{op}} \times \mathcal{B}_{\Delta} \rightarrow \mathcal{S}$, given by $\Gamma \circ [-, -]$

We have the following equivalences:

$$\text{map}_{\mathcal{B}_{\Delta}}(-, (-)^{(-)}) \simeq \text{map}_{\mathcal{B}_{\Delta}}(- \otimes -, -) \simeq \text{map}_{\mathcal{S}_{\Delta}}(-, \text{Fun}_{\mathcal{B}}(-, -))$$

Internally Local

Definition

We say that a map $f : X \rightarrow Y$ of objects in $\mathcal{B}$ is internally local w.r.t. $Z \in \mathcal{B}$ if the canonical map $f^* : [Y, Z] \rightarrow [X, Z]$ is an equivalence.

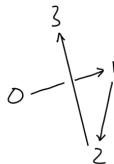
- By Yoneda, this is equivalent to the condition that $f^* : \text{map}_{\mathcal{B}}(A \times Y, Z) \rightarrow \text{map}_{\mathcal{B}}(A \times X, Z)$ is an equivalence for all $A \in \mathcal{B}$.

Spine

The n -spine I^n is defined the simplicial subcomplex

$1 \rightarrow 2 \rightarrow \cdots \rightarrow n$ in Δ^n . It can also be described as the pushout

$$\Delta^1 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^0} \Delta^1.$$

 Δ^3

 I^3

Walking Equivalence

Let E^1 be the pushout

$$\begin{array}{ccc}
 \Delta^1 \sqcup \Delta^1 & \xrightarrow{s^0 \sqcup s^0} & \Delta^0 \sqcup \Delta^0 \\
 \downarrow d^{\{0,2\}} \sqcup d^{\{1,3\}} & & \downarrow \\
 \Delta^3 & \xrightarrow{\quad \quad \quad} & E^1
 \end{array}$$

Take a simplicial object $C : N(\mathbf{\Delta})^{\text{op}} \rightarrow \mathcal{B}$. A map $E^1 \rightarrow C$ is the data of a morphism $f : x \rightarrow y$, left and right inverses $g, h : y \rightarrow x$, and a 3-simplex composing them.

$\mathcal{B}$ -Category

Definition ($\mathcal{B}$ -Category)

A functor $C : N(\mathbf{\Delta})^{\text{op}} \rightarrow \mathcal{B}$ is a $\mathcal{B}$ -category if it is internally local w.r.t. $I^n \hookrightarrow \Delta^n$ (Segal condition) and w.r.t. $E^1 \rightarrow \Delta^0$ (univalence).

- Equivalently, we have equivalences $C_n \rightarrow C_1 \times_{C_0} \cdots \times_{C_0} C_1$ and $C_0 \rightarrow (C_0 \times C_0) \times_{C_1 \times C_1} C_3$
- The Segal condition gives composition which is associative and unital up to homotopy

Objects and Mapping Spaces

- An object of $\mathcal{C}$ in context A is a map $c : A \rightarrow C_0$ for $A \in \mathcal{B}$.
- Given $c, d : A \rightarrow C_0$, define $\text{map}_{\mathcal{C}}(c, d) \in \mathcal{B}/_A$ as the pullback

$$\begin{array}{ccc}
 \text{map}_{\mathcal{C}}(c, d) & \longrightarrow & C_1 \\
 \downarrow & \lrcorner & \downarrow (d^1, d^0) \\
 A & \xrightarrow{(c, d)} & C_0 \times C_0
 \end{array}$$

Univalence

- Two ways objects can be equivalent: $x, y : A \rightarrow C_0$ are in the same “path component” (i.e. equivalent in $\mathcal{B}$) or there is a morphism $f : \Delta^1 \otimes A \rightarrow C$ whose source and target are x, y and which has a left and right inverse
- Consider $d^{\{1,2\}} : \Delta^1 \hookrightarrow \Delta^3$. Induces map $\Delta^1 \otimes A \rightarrow E^1 \otimes A$.
- f is an equivalence iff it factors through $\Delta^1 \otimes A \rightarrow E^1 \otimes A$
- Univalence condition implies maps $E^1 \otimes A \rightarrow C$ correspond to maps $A \rightarrow C$ by factoring through $E^1 \otimes A \rightarrow A$
- So equivalences $\Delta^1 \otimes A \rightarrow C$ correspond to maps $A \rightarrow C$

Univalence (continued)

- $\mathcal{B}$ -groupoid of equivalences between $c, d : A \rightarrow C_0$ is the pullback square

$$\begin{array}{ccc}
 \mathrm{eq}_{\mathcal{C}}(c, d) & \longrightarrow & C_0 \\
 \downarrow & \lrcorner & \downarrow (d_1, d_0) \\
 A & \xrightarrow{(c, d)} & C_0 \times C_0
 \end{array}$$

- Computation shows sections of $\mathrm{eq}_{\mathcal{C}}(c, d)$ are pairs (x, h) where $x : A \rightarrow C_0$ and $h : (c, d) \rightarrow (x, x)$ is an equivalence.
- In other words, equivalences of c, d (in the higher category sense) are equivalences of c, d (in the $\mathcal{B}$ sense)

$\mathcal{B}$ -Groupoids

- A $\mathcal{B}$ -groupoid should be a $\mathcal{B}$ -category where all 1-morphisms are equivalences
- For all $c, d : A \rightarrow C$, $s^0 : \Delta^1 \rightarrow \Delta^0$ should induce equivalence $\text{eq}_C(c, d) = C_0 \times_{C_0 \times C_0} A \rightarrow C_1 \times_{C_0 \times C_0} A = \text{map}_C(c, d)$
- By universality of colimits, s^0 induces equivalence $C_0 \rightarrow C_1$
- By Segal condition, $C_0 \simeq C_n$
- So $\mathcal{B}$ -groupoids are constant simplicial objects of $\mathcal{B}$, i.e. objects of $\mathcal{B}$

Sheaves of ∞ -Categories

- Define $\mathbf{PSh}_{\mathcal{S}}(\mathcal{B}) := \mathbf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathcal{S})$. By Yoneda, $\mathcal{B} \hookrightarrow \mathbf{PSh}_{\mathcal{S}}(\mathcal{B})$
- Taking functor categories gives $\mathcal{B}_{\Delta} \hookrightarrow \mathbf{PSh}_{\mathcal{S}_{\Delta}}(\mathcal{B})$
- Since mapping spaces preserve limits, univalence and Segal condition are preserved, so this restricts to an embedding $\mathbf{Cat}(\mathcal{B}) \hookrightarrow \mathbf{PSh}_{\mathbf{Cat}_{\infty}}(\mathcal{B})$
- Essential image of embedding is $\mathbf{Sh}_{\mathbf{Cat}_{\infty}}(\mathcal{B}) \simeq \mathbf{Cat}(\mathcal{B})$ (limit preserving presheaves)
- Can do this with large categories: $\mathbf{Sh}_{\mathbf{Cat}_{\infty}}(\hat{\mathcal{B}}) \simeq \mathbf{Cat}(\hat{\mathcal{B}})$

$\mathcal{B}$ -Category of Groupoids

- Descent for topoi states $\mathcal{B}_{/_-} : \mathcal{B}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ is sheaf
- By previous slide, this determines a (large) $\mathcal{B}$ -category Ω with the property $\text{Fun}_{\mathcal{B}}(A, \Omega) \simeq \mathcal{B}_{/A}$ for A a $\mathcal{B}$ -groupoid
- In other words, functors from a $\mathcal{B}$ -groupoid A to Ω correspond to maps $X \rightarrow A$ in $\mathcal{B}$
- What about $\text{Fun}_{\mathcal{B}}(C, \Omega)$ for C a $\mathcal{B}$ -category? (more on this soon)

Left Fibrations

Definition

A map $p : C \rightarrow D$ is a left fibration if it is internally right orthogonal to $d^1 : \Delta^0 \hookrightarrow \Delta^1$, i.e. $C^{\Delta^1} \simeq C \times_D D^{\Delta^1}$

$$\begin{array}{ccccc}
 F & \longrightarrow & C^{\Delta^1} & \longrightarrow & D^{\Delta^1} \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 A & \xrightarrow{c} & C & \longrightarrow & D
 \end{array}$$

- Left square pullback implies $F \simeq C_{c/}$
- Big square pullback implies $F \simeq D_{p(c)/}$

Left Fibrations (cont.)

- We have functor $\mathbf{LFib}(- \times -) : \mathcal{B}^{\mathrm{op}} \times \mathbf{Cat}(\mathcal{B})^{\mathrm{op}} \rightarrow \hat{\mathbf{Cat}}_{\infty}$
- Sends $A \times C$ to the ∞ -category of maps $D \rightarrow A \times C$ which are left fibrations
- Fixing a $\mathcal{B}$ -category C , we have sheaf $\mathbf{LFib}(- \times C) \rightarrow \hat{\mathbf{Cat}}_{\infty}$
- This gives associated (large) internal category $\underline{\mathbf{LFib}}_C$ whose global sections are $\mathbf{LFib}(- \times C)$

Grothendieck Construction

Theorem

For any $\mathcal{B}$ -category C , the presheaf $\underline{\mathbf{LFib}}_C$ defines a large $\mathcal{B}$ -category, and $\underline{\mathbf{Fun}}_{\mathcal{B}}(C, \Omega) \simeq \underline{\mathbf{LFib}}_C$.

- Taking global sections, $\mathbf{Fun}_{\hat{\mathcal{B}}}(C, \Omega) \simeq \mathbf{LFib}(C)$
- This equivalence is given by pullback along a universal left fibration $\hat{\Omega} \rightarrow \Omega$
- Can study functors $C \rightarrow \Omega$ by studying left fibrations over C

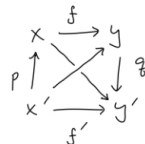
Twisted Diagonal

- There is a map $\epsilon : \mathbf{\Delta} \rightarrow \mathbf{\Delta}$ given by $[n] \mapsto [n]^{\text{op}} \star [n] \cong [2n + 1]$.
- Given a $\mathcal{B}$ -category C , define $\text{Tw}(C) := C \circ \epsilon^{\text{op}}$

$$[0]^{\text{op}} \star [0] : 0 \longrightarrow 0'$$

$$[1]^{\text{op}} \star [1] :$$


$$\text{Tw}(C)_0 : x \xrightarrow{f} y$$

$$\text{Tw}(C)_1 :$$


Mapping Space Functor

- There are two canonical maps $[n]^{\text{op}} \hookrightarrow [n]^{\text{op}} \star [n] \hookleftarrow [n]$
- These induce a map $\text{Tw}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}} \times \mathcal{C}$
- Should view these maps as source and target maps
- This map is a left fibration (Segal condition)
- By the Grothendieck construction, this corresponds to a map $\text{map}_{\mathcal{B}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \Omega$
- Why this notation?

Mapping Space Functor (cont.)

- Take generalized element $(c, d) : A \rightarrow (C^{\text{op}} \times C) \simeq C_0 \times C_0$
- Left fibrations are conservative (right square is Cartesian)
- Outer square is pullback of (c, d) along $\text{Tw}(C) \rightarrow C^{\text{op}} \times C$
- $\text{Tw}_0(C) \rightarrow C_0 \times C_0$ is source and target map (d^1, d^0)

$$\begin{array}{ccccc}
 F & \longrightarrow & \text{Tw}(C)_0 & \longrightarrow & \text{Tw}(C) \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 A & \xrightarrow{(c,d)} & C_0 \times C_0 & \longrightarrow & C^{\text{op}} \times C
 \end{array}$$

- Thus $F \simeq \text{map}_{\mathcal{B}}(c, d)$

Yoneda Map

- We have adjunction $C^{op} \times (-) \dashv [C^{op}, -]$, counit denoted ev
- Denote the transpose of $map_B(-, -)$ by $h : C \rightarrow [C^{op}, \Omega]$
- For clearer notation, $\underline{PSh}(C) := [C^{op}, \Omega]$

Theorem (Yoneda's Lemma)

The following diagram commutes

$$\begin{array}{ccc}
 C^{op} \times \underline{PSh}(C) & \xrightarrow{h \times id} & \underline{PSh}(C)^{op} \times \underline{PSh}(C) \\
 & \searrow ev & \downarrow map_{\underline{PSh}(C)}(-, -) \\
 & & \Omega
 \end{array}$$

Fully Faithful

Definition (Fully Faithful)

A map $f : C \rightarrow D$ is fully faithful if it is internally right orthogonal to $(d^1, d^0) : \Delta^0 \sqcup \Delta^0 \rightarrow \Delta^1$, i.e. $C^{\Delta^1} \simeq (C \times C) \times_{D \times D} D^1$.

- Equivalently, $f : \text{map}_C(c_0, c_1) \rightarrow \text{map}_D(f(c_0), f(c_1))$ is an equivalence for every pair $c_0, c_1 \in C_0$
- Equivalently, $C_1 \simeq (C_0 \times C_0) \times_{D_0 \times D_0} D_1$

Yoneda Embedding

Corollary (Yoneda Embedding)

For any $\mathcal{B}$ -category C , the Yoneda embedding $h : C \rightarrow \underline{\text{PSh}}(C)$ is fully faithful.

- By Yoneda lemma and counit-unit definition of adjoints,

$$\text{map}_{\underline{\text{PSh}}(C)}(-, -) \circ (h^{\text{op}} \times h) \simeq \text{map}_C(-, -)$$

- By this and functorality of twist construction:

$$\begin{array}{ccccc}
 \text{Tw}(C) & \longrightarrow & \text{Tw}(\underline{\text{PSh}}(C)) & \longrightarrow & \hat{\Omega} \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 C^{\text{op}} \times C & \longrightarrow & \underline{\text{PSh}}(C)^{\text{op}} \times \underline{\text{PSh}}(C) & \longrightarrow & \Omega
 \end{array}$$

Yoneda Embedding (cont.)

- Big square and right square are pullbacks, so left square is pullback
- Now take objects (this preserves pullbacks):

$$\begin{array}{ccc}
 C_1 & \xrightarrow{\quad} & \underline{\text{PSh}}(C)_1 \\
 \downarrow & \lrcorner & \downarrow \\
 C_0 \times C_0 & \longrightarrow & \underline{\text{PSh}}(C)_0 \times \underline{\text{PSh}}(C)_0
 \end{array}$$

- Top and bottom maps are Yoneda map, so this is exactly the pullback square we want!

Other Results

- A later paper develops (co)limits, adjoints, and Kan extensions internally
- Future projects: develop internal stable ∞ -category / internal higher algebra
- Use these tools to discuss concepts such as K -theory and noncommutative motives from an internal perspective

Sources

- *Yoneda's Lemma for Internal Higher Categories* (Louis Martini), <https://arxiv.org/abs/2103.17141>
- *Higher Topos Theory* (Jacob Lurie), <https://www.math.ias.edu/~lurie/papers/HTT.pdf>