

Hodge Decomposition Theorem

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All manifolds (M, g) are assumed to be smooth, compact, oriented, Riemannian, and of dimension m unless otherwise stated. These notes are adopted from Warner's book *Foundations of Differentiable Manifolds and Lie Groups* and the lectures 25 and 26 of Professor Zuoqin Wang's (USTC) Spring 2016 "Riemannian Geometry" course. I will use Einstein summation notation.

Prerequisites / Review

Definition 1: (de Rham cohomology $\mathbb{R}$ -vector spaces)

Let $\Omega^p(M)$ be the space of smooth p -forms on M . The differential $d_p : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ is defined locally for $\varphi = f dx^{i_1} \wedge \dots \wedge dx^{i_p}$ by $d_p \varphi = \sum_{j=1}^p \frac{\partial f}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$. We will often denote the differential by d when the degree is clear. For combinatorial reasons, $d^2 = d \circ d = 0$, thus we have a chain complex:

$$0 \rightarrow \Omega^0(M) \xrightarrow{d_0} \dots \xrightarrow{d_{n-2}} \Omega^{n-1}(M) \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} \Omega^{n+1}(M) \xrightarrow{d_{n+1}} \dots$$

The de Rham cohomology $\mathbb{R}$ -vector spaces are then defined as $H^n(M; \mathbb{R}) := \ker(d_n) / \text{im}(d_{n-1})$. Sometimes we will denote $Z^n(M) := \ker(d_n)$ and $B^n(M) := \text{im}(d_{n-1})$.

The differential d has the property that if $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$, then $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$.

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Definition 2: (hodge star operator)

The Hodge star is a map $\star : \Omega^p(M) \rightarrow \Omega^{m-p}(M)$ characterized by $\alpha \wedge \star \beta = (\alpha, \beta) \omega_g$ for all p -forms α , where $\omega_g = \sqrt{\det g_{ij}} dx^1 \wedge \dots \wedge dx^m$ is the volume form induced by the metric g and $(-, -) : \Omega^p(M) \otimes_{\mathbb{R}} \Omega^p(M) \rightarrow \mathbb{R}$ defined on one forms by:

$$(\omega_i dx^i, \eta_j dx^j) = g^{ij} \omega_i \eta_j$$

And extend by demanding the $\{dx^{i_1} \wedge \dots \wedge dx^{i_k} : i_1 < \dots < i_k\}$ to be orthonormal.

What does this look like more explicitly?

The Hodge star is $\mathbb{R}$ -linear and satisfies $\star \star \alpha = (-1)^{p(m-p)} \alpha$ when $\alpha \in \Omega^p(M)$.

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Definition 3: (Laplace-Beltrami operator)

Define the codifferential $\delta := (-1)^{m(p+1)+1} \star d \star$; this is a map $\Omega^p(M) \rightarrow \Omega^{p-1}(M)$. The Laplace-Beltrami operator $\Delta : \Omega^p(M) \rightarrow \Omega^p(M)$ is given by $\Delta := d\delta + \delta d$, where d is the differential from Definition 1.

Definition 4: (inner product on $\Omega^p(M)$)

Given $\alpha, \beta \in \Omega^p(M)$, define $\langle \alpha, \beta \rangle := \int_M \alpha \wedge \star \beta$. We can extend to $\Omega^*(M) := \bigoplus_{n \in \mathbb{N}} \Omega^n(M)$ by requiring $\langle \eta, \xi \rangle = 0$ for $\eta \in \Omega^p(M), \xi \in \Omega^q(M), p \neq q$.

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Basic Properties of Δ

Proposition 1

The operators δ and d are adjoint w.r.t. the inner product of Definition 4.

Proof:

Using linearity of the inner product, we can consider the case that $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$. Then:

$$d(\alpha \wedge \star \beta) = d\alpha \wedge \star \beta + (-1)^{p-1} \alpha \wedge d\star \beta$$

Note that:

$$\star \delta = \star (-1)^{m(p+1)+1} \star d\star = (-1)^{m(p+1)+1} \star \star d\star$$

Since $\beta \in \Omega^p(M)$, $\star \beta \in \Omega^{m-p}(M)$ and $d\star \beta \in \Omega^{m-p+1}(M)$ With this in mind, we have:

$$\star \star d\star \beta = (-1)^{(m-p+1)(m-(m-p+1))} d\star \beta = (-1)^{(m-p+1)(p-1)}$$

And note:

$$\begin{aligned} (m-p+1)(p-1) + m(p+1) + 1 + p - 1 &= mp - m - p^2 + p + p - 1 + mp + m + 1 + p - 1 \\ &= 2mp - p^2 + p + 2p - 1 \equiv 1 \pmod{2} \end{aligned}$$

Putting these observations together:

$$(-1)^{p-1} \alpha \wedge d\star \beta = -\alpha \wedge \star \delta \beta$$

Thus:

$$d(\alpha \wedge \star \beta) = d\alpha \wedge \star \beta - \alpha \wedge \star \delta \beta$$

Integrating both sides and noting that $d(\alpha \wedge \star \beta) \in \Omega^{m+1}(M) = 0$:

$$0 = \int_M d(\alpha \wedge \star \beta) = \int_M d\alpha \wedge \star \beta - \int_M \alpha \wedge \star \delta \beta$$

Therefore:

$$\langle d\alpha, \beta \rangle = \int_M d\alpha \wedge \star \beta = \int_M \alpha \wedge \star \delta \beta = \langle \alpha, \delta \beta \rangle$$

By symmetry of the inner product, we also have $\langle \delta \beta, \alpha \rangle = \langle \beta, d\alpha \rangle$.

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Corollary 1

The operator Δ is self-adjoint w.r.t. the inner product of Definition 4.

Proof:

By linearity of the inner product, we can consider $\alpha, \beta \in \Omega^p(M)$. Using linearity of the maps at hand and Proposition 1:

$$\begin{aligned} \langle \Delta \alpha, \beta \rangle &= \langle (d\delta + \delta d)\alpha, \beta \rangle = \langle d\delta \alpha, \beta \rangle + \langle \delta d\alpha, \beta \rangle \\ &= \langle \delta \alpha, \delta \beta \rangle + \langle d\alpha, d\beta \rangle = \langle \alpha, d\delta \beta \rangle + \langle \alpha, \delta d\beta \rangle \\ &= \langle \alpha, (d\delta + \delta d)\beta \rangle = \langle \alpha, \Delta \beta \rangle \end{aligned}$$

Proposition 2

$\Delta\alpha = 0$ iff $d\alpha = \delta\alpha = 0$.

Proof:

Obviously if $d\alpha = \delta\alpha = 0$ then $\Delta\alpha = 0$. On the other hand, if $\Delta\alpha = 0$, then:

$$0 = \langle \Delta\alpha, \alpha \rangle = \langle (d\delta + \delta d)\alpha, \alpha \rangle = \langle d\delta\alpha, \alpha \rangle + \langle \delta d\alpha, \alpha \rangle = \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle$$

Which by positive-definiteness of inner products implies $\delta\alpha = d\alpha = 0$.

Proposition 3

We have $\Delta \star = \star \Delta$.

Proof:

This is just unpacking definitions. Let $\omega \in \Omega^p(M)$. Evaluating, we see:

$$\star d\delta\omega = (-1)^{m(p+1)+1} \star d \star d \star \omega = \delta d \star \omega$$

$$\star \delta d\omega = (-1)^{m(p+1)+1} \star \star d \star d\omega = (-1)^{m(p+1)+1} (-1)^{p(m-p)} d \star d\omega = d\delta \star \omega$$

Thus $\star d\delta = \delta d \star$ and $\star \delta d = d\delta \star$. Adding together, $\star \Delta = \star d\delta + \star \delta d = \delta d \star + d\delta \star = \Delta \star$.

Definition 5: (harmonic p-forms)

A p -form $\alpha \in \Omega^p(M)$ is said to be harmonic if $\Delta\alpha = 0$. Let $\mathcal{H}^p(M)$ be the $\mathbb{R}$ -vector space of harmonic p -forms on M .

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Some Definitions and Useful Theorems

Definition 6: (weak solution)

Given $\alpha \in \Omega^p(M)$, we say that a bounded linear functional $\ell : \Omega^i(M) \rightarrow \mathbb{R}$ is a weak solution to $\Delta\omega = \alpha$ if $\ell(\Delta\varphi) = \langle \alpha, \varphi \rangle$ for all $\varphi \in \Omega^p(M)$.

Theorem 1 (Regularity Theorem)

Let $\alpha \in \Omega^p(M)$, and let ℓ be a weak solution of $\Delta\omega = \alpha$. Then there exists $\omega' \in \Omega^p(M)$ such that $\ell(\beta) = \langle \omega, \beta \rangle$ for every $\beta \in \Omega^p(M)$.

This implies $\langle \alpha, \beta \rangle = \ell(\Delta\beta) = \langle \omega', \Delta\beta \rangle = \langle \Delta\omega', \beta \rangle$ for any $\beta \in \Omega^p(M)$ and thus $\alpha = \Delta\omega'$.

Theorem 2

Let $\{\alpha_n \in \Omega^p(M)\}$ be such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n and for some constant $c > 0$. Then there is a Cauchy subsequence in $\Omega^p(M)$.

Hodge Decomposition Theorem

Note that Proposition 2 implies every harmonic p -form is closed and thus determines a class in $H^p(M; \mathbb{R})$. The Hodge Decomposition Theorem says that each class in $H^p(M; \mathbb{R})$ uniquely determines an element of $\mathcal{H}^p(M)$.

Hodge Decomposition Theorem

Let (M, g) be a closed, oriented, Riemannian manifold. Then for each p , $\mathcal{H}^p(M)$ is finite-dimensional, and there is an orthogonal decomposition $\Omega^p(M) = \Delta(\Omega^p(M)) \oplus \mathcal{H}^p(M)$.

Proof:

Suppose f.s.o.c. $\mathcal{H}^p(M)$ is infinite-dimensional. Then there exists an o.n. sequence $\{\alpha_n\}$ in $\mathcal{H}^p(M)$. Since the α_n are harmonic, $\|\Delta\alpha_n\| = \|0\| = 0$ for all n . By normality, $\|\alpha_n\| = 1$, thus by Theorem 2 there exists a Cauchy subsequence α_{n_k} . But this is impossible since for any $n, m \in \mathbb{N}$, $\|\alpha_n - \alpha_m\|^2 = \|\alpha_n\|^2 + 2\langle\alpha_n, \alpha_m\rangle + \|\alpha_m\|^2 = 1 + 0 + 1 = 2$ by orthogonormality. Thus $\mathcal{H}^p(M)$ is finite-dimensional.

From basic linear algebra, we have a decomposition $\Omega^p(M) = \mathcal{H}^p(M) \oplus \mathcal{H}^p(M)^\perp$, the latter space being the orthogonal complement. Let $H : \Omega^p(M) \rightarrow \mathcal{H}^p(M)$ be the obvious projection map.

(1) $\Delta(\Omega^p) \subseteq \mathcal{H}^p(M)^\perp$

We will show $\Delta(\Omega^p) := \Delta(\Omega^p(M)) = \mathcal{H}^p(M)^\perp$. First, note that for any $\alpha \in \mathcal{H}^p(M)$ and $\omega \in \Omega^p(M)$:

$$\langle \Delta\omega, \alpha \rangle = \langle \omega, \Delta\alpha \rangle = \langle \omega, 0 \rangle = 0$$

Thus $\Delta(\Omega^p) \subseteq \mathcal{H}^p(M)^\perp$.

(2) There exists $c > 0$ s.t. $\|\beta\| \leq c\|\Delta\beta\| \quad \forall \beta \in \mathcal{H}^p(M)^\perp$

Suppose no such constant exists. Then there is a sequence $\{\beta_j\}$ in $\mathcal{H}^p(M)^\perp$ such that $\|\Delta\beta_j\| \rightarrow 0$; normalizing we can assume $\|\beta_j\| = 1$ for each j . By Theorem 6.6, there is a Cauchy subsequence $\beta'_k := \beta_{j_k}$. Define $\ell : \Omega^p(M) \rightarrow \mathbb{R}$ by $\ell(\psi) = \lim_{k \rightarrow \infty} \langle \beta'_k, \psi \rangle$. By Cauchy-Schwarz:

$$|\ell(\psi)| = \left| \lim_{k \rightarrow \infty} \langle \beta'_k, \psi \rangle \right| = \lim_{k \rightarrow \infty} |\langle \beta'_k, \psi \rangle| \leq \lim_{k \rightarrow \infty} \|\beta'_k\| \|\psi\| \leq \|\psi\|$$

Thus ℓ is a bounded operator. Also note:

$$\ell(\Delta\varphi) = \lim_{k \rightarrow \infty} \langle \beta'_k, \Delta\varphi \rangle = \lim_{k \rightarrow \infty} \langle \Delta\beta'_k, \varphi \rangle = 0$$

By Cauchy-Schwarz and since $\|\Delta\beta'_k\| \rightarrow 0$. Thus by Theorem 1, there exists $\beta \in \Omega^p(M)$ such that $\ell(\psi) = \langle \beta, \psi \rangle$. Since $\beta'_k \rightarrow \beta$ weakly and since $\beta \in \Omega^p(M)$, $\beta'_k \rightarrow \beta$ w.r.t. $\|\cdot\|$. Thus $\|\beta\| = \lim_{k \rightarrow \infty} \|\beta'_k\| = 1$, so $\beta \neq 0$. We also know $\eta \in \mathcal{H}^p(M)^\perp$ since each $\beta_j \in \mathcal{H}^p(M)^\perp$ and thus for any $\alpha \in \mathcal{H}^p(M)$:

$$\langle \beta, \alpha \rangle = \left\langle \lim_{k \rightarrow \infty} \beta'_k, \alpha \right\rangle = \lim_{k \rightarrow \infty} \langle \beta'_k, \alpha \rangle = 0$$

But $\Delta\beta = \Delta(\lim_{k \rightarrow \infty} \beta'_k) = \lim_{k \rightarrow \infty} \Delta\beta'_k = 0$ since $\|\Delta\beta'_k\| \rightarrow 0$, thus $\beta \in \mathcal{H}^p(M)$. But this contradicts $\mathcal{H}^p(M) \cap \mathcal{H}^p(M)^\perp = \{0\}$, thus such a c must exist.

(3) $\mathcal{H}^p(M) \subseteq \Delta(\Omega^p)$

Take $\alpha \in \mathcal{H}^p(M)^\perp$ and define ℓ on $\Delta(\Omega^p)$ by $\ell(\Delta\varphi) = \langle \alpha, \varphi \rangle, \varphi \in \Omega^p(M)$. To show well-definedness, if $\Delta\varphi = \Delta\varphi'$, then $\varphi - \varphi' \in \mathcal{H}^p(M)$ and so $0 = \langle \alpha, \varphi - \varphi' \rangle \implies \langle \alpha, \varphi \rangle = \langle \alpha, \varphi' \rangle$. Also note that ℓ is bounded on $\Delta(\Omega^p)$, for given any $\varphi \in \Omega^p(M)$, define $\psi := \varphi - H(\varphi)$; by definition of H we have $\psi \in \mathcal{H}^p(M)^\perp$ and $\Delta\psi = \Delta\varphi$. Using these observations, the result from above, and Cauchy-Schwarz:

$$|\ell(\Delta\varphi)| = |\ell(\Delta\psi)| = |\langle \alpha, \psi \rangle| \leq \|\alpha\| \|\psi\| \leq c\|\alpha\| \|\Delta\psi\| = c\|\alpha\| \|\Delta\varphi\|$$

Because of these facts, by the Hahn-Banach theorem ℓ extends to a bounded linear functional on $\Omega^p(M)$, so by Theorem 1 there exists $\omega \in \Omega^p(M)$ with $\Delta\omega = \alpha$. Thus $\alpha \in \Delta(\Omega^p)$, so $\mathcal{H}^p(M)^\perp \subseteq \Delta(\Omega^p)$. With the other inclusion, this proves the claim. ■

Construction of Green's Operator

Note this theorem implies $\Delta : \mathcal{H}^p(M)^\perp \rightarrow \mathcal{H}^p(M)^\perp$ is bijective. Thus we can define a map $G : \Omega^p(M) \rightarrow \mathcal{H}^p(M)^\perp$ by $G := (\Delta|_{\perp})^{-1} \circ \pi_\perp$, where $\pi_\perp : \Omega^p(M) \rightarrow \mathcal{H}^p(M)^\perp$ is given by $a \mapsto a - H(a)$ and $\Delta|_{\perp} := \Delta|_{\mathcal{H}^p(M)^\perp}$.

Proposition 4

G commutes with any linear operator $T : \Omega^p(M) \rightarrow \Omega^q(M)$ that commutes with Δ .

Proof:

Because $T\Delta = \Delta T$, if $\varphi \in \mathcal{H}^p(M)$, then $\Delta T(\varphi) = T(\Delta\varphi) = T(0) = 0$, so $T(\mathcal{H}^p(M)) \subseteq \mathcal{H}^q(M)$. Furthermore, since $\Delta(\Omega^p) = \mathcal{H}^p(M)^\perp$, $T(\Delta(\Omega^p)) = \Delta(T(\Omega^p)) \subseteq \Delta(\Omega^q)$, thus T restricts to a map $\mathcal{H}^p(M)^\perp \rightarrow \mathcal{H}^q(M)^\perp$. From these facts, $T(H(\alpha)) \in \mathcal{H}^p(M)$ and $T(\alpha - H(\alpha)) \in \mathcal{H}^p(M)^\perp$, thus $T(\alpha - H(\alpha)) = T(\alpha) - H(T(\alpha))$ by the direct sum decomposition. In particular, this implies:

$$T \circ \pi_\perp = \pi_\perp \circ T$$

Additionally, $T\Delta|_{\perp} = \Delta|_{\perp} T$ since $T\Delta = \Delta T$ and $T(\mathcal{H}^p(M)^\perp) \subseteq \mathcal{H}^q(M)^\perp$. Since $\Delta|_{\perp}$ is invertible, we have $(\Delta|_{\perp})^{-1}T = T(\Delta|_{\perp})^{-1}$. Combining with our earlier statement:

$$TG = T(\Delta|_{\perp})^{-1}\pi_\perp = (\Delta|_{\perp})^{-1}T\pi_\perp = (\Delta|_{\perp})^{-1}\pi_\perp T = GT$$
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Theorem 3

Each de Rham cohomology class on a compact oriented Riemannian manifold M contains a unique harmonic representative.

Proof:

Let $\alpha \in Z^p(M)$. From the Hodge decomposition theorem and the definition of Green's operator:

$$\alpha = \Delta G(\alpha) + H(\alpha) = (d\delta + \delta d)G(\alpha) + H(\alpha) = d\delta G(\alpha) + \delta dG(\alpha) + H(\alpha)$$

Since G commutes with d and $\alpha \in Z^p(M)$, $\delta dG(\alpha) = \delta Gd(\alpha) = \delta G(0) = 0$, thus:

$$\alpha - H(\alpha) = d(\delta G(\alpha))$$

So that α and $H(\alpha)$ are cohomologous and thus define the same equivalence class in $H^p(M; \mathbb{R})$. Thus every class in $H^p(M; \mathbb{R})$ contains a harmonic representative.

On the other hand, suppose $[\alpha_1] = [\alpha_2] \in H^p(M; \mathbb{R})$ and $\alpha_1, \alpha_2 \in \mathcal{H}^p(M)$. Then there exists $\beta \in \Omega^{p-1}(M)$ with $\alpha_1 - \alpha_2 = d\beta$ and $\Delta(\alpha_1 - \alpha_2) = 0$. Now note that by Proposition 2:

$$\langle d\beta, \alpha_1 - \alpha_2 \rangle = \langle \beta, \delta(\alpha_1 - \alpha_2) \rangle = \langle \beta, 0 \rangle = 0$$

Thus $d\beta$ and $\alpha_1 - \alpha_2$ are orthogonal, so $d\beta = 0$ or $\alpha_1 - \alpha_2 = 0$; in either case we find $\alpha_1 = \alpha_2$, thus the harmonic representative from each class is unique. ■

A Topological Application

We have a bilinear map $H^p(M; \mathbb{R}) \otimes_{\mathbb{R}} H^{m-p}(M; \mathbb{R}) \rightarrow \mathbb{R}$ defined by:

$$([\varphi], [\psi]) \mapsto \int_M \varphi \wedge \psi$$

If $[\varphi] = [\varphi']$, then $\varphi - \varphi' = d\beta$ for some $\beta \in \Omega^{p-1}(M)$ and thus $(\varphi - \varphi') \wedge \psi = d\beta \wedge \psi = d(\beta \wedge \psi) - (-1)^{p-1} \beta \wedge d\psi = d(\beta \wedge \psi)$ since $\psi \in Z^{m-p}(M)$. Therefore by Stokes's Theorem:

$$\int_M (\varphi - \varphi') \wedge \psi = \int_M d(\beta \wedge \psi) = 0 \implies \int_M \varphi \wedge \psi = \int_M \varphi' \wedge \psi$$

Thus the mapping is well-defined in the first argument, A similar argument shows well-definedness in the second argument.

Theorem

Let M be a closed, orientable, smooth manifold. The bilinear function above is non-singular and thus determines an isomorphism:

$$H^{m-p}(M; \mathbb{R}) \cong \text{Hom}_{\mathbb{R}}(H^p(M; \mathbb{R}), \mathbb{R}) =: H^p(M; \mathbb{R})^\vee$$

Proof:

Using a partition of unity, we can equip M with a Riemannian metric and apply the results of the Hodge decomposition theorem. Let $[\varphi] \in H^p(M; \mathbb{R})$ be nonzero; we wish to find some $[\psi] \in H^{m-p}(M; \mathbb{R})$ such that $\int_M \varphi \wedge \psi \neq 0$. Without loss of generality, we may assume φ is harmonic; since $[\varphi] \neq 0$, $\varphi \neq 0$. By Proposition 3, $\Delta(\star \varphi) = (\Delta \star) \varphi = (\star \Delta) \varphi = \star(\Delta \varphi) = 0$, thus $\star \varphi$ is harmonic and so represents a class in $H^{m-p}(M; \mathbb{R})$. Then consider $(\varphi, \star \varphi)$ under the bilinear map from above:

$$(\varphi, \star \varphi) \mapsto \int_M \varphi \wedge \star \varphi = \|\varphi\|^2 \neq 0$$

Where the last statement follows since $\varphi \neq 0$. Thus the pairing is nonsingular, so by linear algebra the map $H^{m-p}(M; \mathbb{R}) \rightarrow H^p(M; \mathbb{R})^\vee$ given by $\varphi \mapsto (-, \varphi)$ is an isomorphism. Also note that since the de Rham cohomology ■