

Reference Sheet of Categories

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Ab-category (preadditive category)

A category $\mathcal{A}$ in which every homset $\text{Hom}_{\mathcal{A}}(A, B)$ has the structure of an abelian group and composition distributes over addition, so that given:

$$A \xrightarrow{f} B \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{g'} \end{array} C \xrightarrow{h} D$$

We have $h(g + g')f = hgf + hg'f$.

An **Ab**-category can also be viewed as a category enriched over the monoidal category **Ab**.

Abelian category

An **additive category** $\mathcal{A}$ such that every morphism has a kernel and cokernel, every monic arrow is a kernel, and every epi arrow a cokernel.

An abelian subcategory $\mathcal{B}$ is a subcategory of $\mathcal{A}$ which is abelian and such that exact sequences in $\mathcal{B}$ are exact in $\mathcal{A}$.

What is an example of a subcategory of an abelian category which is not abelian?

Exercise: Show that this is equivalent to the canonical map $\text{coim}(f) \rightarrow \text{im}(f)$ being an isomorphism. Show this is equivalent to every monic arrow is the kernel of its cokernel and every epi arrow the cokernel of its kernel.

Additive category

An **Ab**-category $\mathcal{A}$ with a zero object and product $A \times B$ for each pair $A, B \in \text{Ob}(\mathcal{A})$.

Exercise: Show that in an additive category $A \times B \cong A \oplus B$.

Comma category $\mathcal{C}/d$ of objects over d

Let $\mathcal{C}$ be a category and $d \in \text{Ob}(\mathcal{C})$. Denoted $\mathcal{C}/d$, dual $d \setminus \mathcal{C}$.

Objects: Maps $f : c \rightarrow d$ in $\mathcal{C}$.

Morphisms: Map from $f : c \rightarrow d$ to $f' : c' \rightarrow d$ is $h \in \text{Hom}_{\mathcal{C}}(c, c')$ s.t. $f = f'h$.

Comma category over functor

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Denoted F/d , dual $d \setminus F$.

Objects: Object of F/d for some $d \in \mathcal{D}$.

Morphism: $\left(c, F(c) \xrightarrow{f} d\right) \rightarrow \left(c', F(c') \xrightarrow{f'} d'\right)$ pairs of maps $c \xrightarrow{g} c', d \xrightarrow{h} d'$ s.t.:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(g)} & F(c') \\ f \downarrow & & \downarrow f' \\ d & \xrightarrow{h} & d' \end{array}$$

Comma category over object

Let $F : C \rightarrow D$ be a functor and $d \in \text{Ob}(D)$. Denoted F/d , dual $d \setminus F$.

Objects: Pairs (c, f) with $c \in \text{Ob}(C)$ and $f \in \text{Hom}_D(F(c), d)$.

Morphisms: $(c, f) \rightarrow (c', f')$ is morphism $h \in \text{Hom}_C(c, c')$ s.t.:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(h)} & F(c') \\ & \searrow f & \swarrow f' \\ & d & \end{array}$$

Consider $(E, *_E) \xrightarrow{f} (B, *_B)$ in $\mathbf{Top}_*$. Then $*_B \setminus \Pi f$ functions the same as the homotopy fiber $F(f)$. Here $\Pi : \mathbf{Top}_* \rightarrow \mathbf{Grpd}$ is the fundamental groupoid functor.

When $F = \text{Id}$ is the identity functor, F/d is the comma category C/d (and $d \setminus F$ is $d \setminus C$).

Enriched category

A category enriched over a **monoidal category** $(V, \otimes, 1, a, \lambda, \rho)$ is a category C with the following data:

- (i) A class $\text{Ob}(C)$.
- (ii) For each $(a, b) \in \text{Ob}(C) \times \text{Ob}(C)$, a hom object $C(a, b) \in \text{Ob}(V)$.
- (iii) For each $(a, b, c) \in \text{Ob}(C)^3$ a composition morphism $\circ_{a,b,c} : C(b, c) \otimes C(a, b) \rightarrow C(a, c)$.
- (iv) For each $a \in \text{Ob}(C)$ a morphism $j_a : 1 \rightarrow C(a, a)$

Satisfying the following axioms:

- (1) Associative:

$$\begin{array}{ccc} (C(c, d) \otimes C(b, c)) \otimes C(a, b) & \xrightarrow{a_{C(c, d), C(b, c), C(a, b)}} & C(c, d) \otimes (C(b, c) \otimes C(a, b)) \\ \downarrow \circ_{b, c, d} \otimes \text{id}_{C(a, b)} & & \downarrow \text{id}_{C(c, d)} \otimes \circ_{a, b, c} \\ C(b, d) \otimes C(a, b) & & C(c, d) \otimes C(a, c) \\ \searrow \circ_{a, b, d} & & \swarrow \circ_{a, c, d} \\ & C(a, d) & \end{array}$$

- (2) Unital:

$$\begin{array}{ccccc} C(b, b) \otimes C(a, b) & \xrightarrow{\circ_{a, b, b}} & C(a, b) & \xleftarrow{\circ_{a, a, b}} & C(a, b) \otimes C(a, a) \\ j_b \otimes \text{id}_{C(a, b)} \uparrow & & \nearrow \lambda_{C(a, b)} & & \nwarrow \rho_{C(a, b)} \\ 1 \otimes C(a, b) & & C(a, b) & & C(a, b) \otimes 1 \\ & & & & \uparrow \text{id}_{C(a, b)} \otimes j_a \end{array}$$

A preadditive category is a category enriched in $(\mathbf{Ab}, \otimes)$. A 2-category is a category enriched in $(\mathbf{Cat}, \times)$.

I ∫ X

Let I be a small category and $X : I \rightarrow \mathbf{Set}$ a functor. Denoted $I \int X$.

Objects: Pairs (i, x) with $i \in \mathbf{Ob}(I)$ and $x \in X(i)$.

Morphisms: $(i, x) \rightarrow (i', x')$ pair of maps $f \in \mathbf{Hom}_I(i, i')$ and $\phi \in \mathbf{Hom}_{\mathbf{Set}}(X(f)(x), x')$ with $x' \in X(i')$

The “Translation Category” below is a special case of this.

Inverse image

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $d \in \mathbf{Ob}(\mathcal{D})$. Denoted $F^{-1}(d)$.

Objects: $c \in \mathbf{Ob}(\mathcal{C})$ s.t. $F(c) = d$.

Morphisms: $h \in \mathbf{Maps}(\mathcal{C})$ s.t. $F(h) = \text{id}_d$.

This functor is isomorphic to the full subcategory of F/d consisting of pairs $(c, F(c) \xrightarrow{=} d)$ or the full subcategory of $d \backslash F$ of pairs $(d \xrightarrow{=} F(c), c)$.

Localization

Let $\mathcal{C}$ be a category and S a collection of morphisms. The localization of $\mathcal{C}$ w.r.t. S is a category $S^{-1}\mathcal{C}$ and functor $q : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ such that $q(s)$ is an isomorphism for all $s \in S$, and any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ sending maps in S to isomorphisms factors uniquely through q . In other, words, there is a unique arrow such that the following commutes

$$\begin{array}{ccc} & S^{-1}\mathcal{C} & \\ q \nearrow & & \searrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

A collection of morphisms S in $\mathcal{C}$ is called a *multiplicative system* if the following axioms are satisfied.

- (1) S is closed under composition and identities.
- (2) (Ore condition) If $t : Z \rightarrow Y$ is in S , then for each $g : X \rightarrow Y$ in $\mathcal{C}$, we can find some object W and maps $s, f : W \rightarrow X, Z$ (resp) such that $s \in S$ and the following commutes

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ s \downarrow & & \downarrow t \\ X & \xrightarrow{g} & Y \end{array}$$

- (3) (cancellation) Given $f, g : X \rightarrow Y$ in $\mathcal{C}$, the following are equivalent

- (a) $sf = sg$ for some $s : Z \rightarrow Y$, $s \in S$
- (b) $ft = gt$ for some $t : Z \rightarrow X$, $t \in S$

We can construct $S^{-1}\mathcal{C}$ as follows.

Let $\text{Ob}(S^{-1}C) = \text{Ob}(C)$. Let fs^{-1} denote the chain $X \xleftarrow{s} X_1 \xrightarrow{f} Y$, where $s \in S$. Then fs^{-1} is equivalent to gt^{-1} if there is some $X \leftarrow X_3 \rightarrow Y$ with the left arrow in S such that the following commutes

$$\begin{array}{ccccc}
 & & X_1 & & \\
 & s \swarrow & \uparrow & \searrow f & \\
 X & \xleftarrow{\quad} & X_3 & \xrightarrow{\quad} & Y \\
 & \nwarrow t & \downarrow & \nearrow g & \\
 & & X_2 & &
 \end{array}$$

Then the morphisms of $S^{-1}C$ are chains modulo the equivalence relation. Composition is defined as follows. Given $X \leftarrow X' \xrightarrow{g} Y$ and $Y \leftarrow Y' \xrightarrow{t} Z$, by the Ore condition there exists a diagram

$$\begin{array}{ccccc}
 & & W & \longrightarrow & Y' & \longrightarrow & Z \\
 & & \downarrow s & & \downarrow t & & \\
 X & \longleftarrow & X' & \xrightarrow{g} & Y & &
 \end{array}$$

Then the composition is $X \leftarrow X' \xleftarrow{s} W \rightarrow Y' \rightarrow Z$ (note that $s \in S$).

Monoidal category

A monoidal category is a category C equipped with the following structure.

- (i) A bifunctor $\otimes : C \times C \rightarrow C$.
- (ii) An object $1 \in \text{Ob}(C)$ called the unit/tensor object.
- (iii) A natural isomorphism $a : ((-) \otimes (-)) \otimes (-) \rightarrow (-) \otimes ((-) \otimes (-))$ called the associator.
- (iv) A natural isomorphism $\lambda : 1 \otimes (-) \rightarrow (-)$ called the left unitor.
- (v) A natural isomorphism $\rho : (-) \otimes 1 \rightarrow (-)$ called the right unitor.

Which satisfies the following axioms.

- (1) Triangle identity:

$$\begin{array}{ccc}
 (x \otimes 1) \otimes y & \xrightarrow{a_{x,1,y}} & x \otimes (1 \otimes y) \\
 \rho_x \otimes 1 \searrow & & \swarrow \lambda \otimes \rho_y \\
 & x \otimes y &
 \end{array}$$

- (2) Pentagon identity:

$$\begin{array}{ccc}
& (w \otimes x) \otimes (y \otimes z) & \\
a_{w \otimes x, y, z} \nearrow & & \searrow a_{w, x, y \otimes z} \\
((w \otimes x) \otimes y) \otimes z & & w \otimes (x \otimes (y \otimes z)) \\
a_{w, x, y} \otimes \text{id}_z \downarrow & & \uparrow \text{id}_w \otimes a_{x, y, z} \\
(w \otimes (x \otimes y)) \otimes z & \xrightarrow{a_{w, x \otimes y, z}} & w \otimes ((x \otimes y) \otimes z)
\end{array}$$

Some examples include $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$, $(\mathbf{Cat}, \times)$.

Quotient abelian category

Let $\mathcal{A}$ be an **abelian category** and $\mathcal{B}$ a **serre subcategory**. A morphism f in $\mathcal{A}$ is a $\mathcal{B}$ -iso if both $\ker f$ and $\text{coker } f$ are in $\mathcal{B}$. The quotient category is denoted $\mathcal{A}/\mathcal{B}$.

Objects: Those of $\mathcal{A}$.

Morphisms: A morphism $A_1 \rightarrow A_2$ is an equivalence class of diagrams of the form:

$$A_1 \xleftarrow{f} A' \xrightarrow{g} A_2, \text{ where } f \text{ is a } \mathcal{B}\text{-iso}$$

Where $A_1 \leftarrow A' \rightarrow A_2$ is equivalent to $A_1 \leftarrow A'' \rightarrow A_2$ iff there is a commutative diagram:

$$\begin{array}{ccccc}
& & A' & & \\
& \swarrow & \uparrow & \searrow & \\
A_1 & \xleftarrow{\quad} & A & \xrightarrow{\quad} & A_2 \\
& \swarrow & \downarrow & \searrow & \\
& & A'' & &
\end{array}$$

Where $A' \leftarrow A \rightarrow A''$ are $\mathcal{B}$ -isos. The composition of $A_1 \xleftarrow{f} A' \xrightarrow{g} A_2$ with $A_2 \xleftarrow{f'} A''' \xrightarrow{h} A_3$ is given by $A_1 \xleftarrow{f} A' \xleftarrow{\quad} A \rightarrow A''' \xrightarrow{h} A_3$, where A is the pullback:

$$\begin{array}{ccc}
A & \xrightarrow{\quad} & A' \\
\downarrow & & \downarrow g \\
A''' & \xrightarrow{f'} & A_2
\end{array}$$

Exercise: Show that $\mathcal{A}/\mathcal{B}$ is abelian and the quotient functor $\text{loc} : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{B}$ is exact.

Quotient category

Let $\mathcal{C}$ be a category. A congruence relation on $\mathcal{C}$ is, for each pair $X, Y \in \text{Ob}(\mathcal{C})$, an equivalence relation $R_{X,Y}$ on $\text{Hom}_{\mathcal{C}}(X, Y)$ which respects composition, i.e., if $f_1 R_{X,Y} f_2$ and $g_1 R_{Y,Z} g_2$, then $g_1 f_1 R_{X,Z} g_2 f_2$. Then the quotient category is denoted $\mathcal{C}/R$.

Objects: Those of $\mathcal{C}$.

Morphisms: $\text{Hom}_{\mathcal{C}/R}(X, Y) := \text{Hom}_{\mathcal{C}}(X, Y)/R$.

Serre subcategory

An abelian subcategory $\mathcal{B}$ of a subcategory $\mathcal{A}$ which is closed under subobjects, quotients, and extensions, i.e., if $0 \rightarrow B \rightarrow C \rightarrow D \rightarrow 0$ is exact in $\mathcal{A}$, then $C \in \mathcal{B} \iff B, D \in \mathcal{B}$.

Symmetric monoidal category

A symmetric monoidal category $\mathcal{C}$ is a **monoidal category** equipped with a natural transformation $s_{ab} : a \otimes b \rightarrow b \otimes a$ satisfying the following axioms:

(1) Unit coherence

$$\begin{array}{ccc} x \otimes 1 & \xrightarrow{s_{x1}} & 1 \otimes x \\ \rho_x \searrow & & \swarrow \lambda_x \\ & x & \end{array}$$

(2) Associativity coherence

$$\begin{array}{ccc} (x \otimes y) \otimes z & \xrightarrow{s_{xy} \otimes \text{id}_z} & (y \otimes x) \otimes z \\ a_{xyz} \downarrow & & \downarrow a_{yxz} \\ x \otimes (y \otimes z) & & y \otimes (x \otimes z) \\ s_{x,y \otimes z} \downarrow & & \downarrow \text{id}_y \otimes s_{xz} \\ (y \otimes z) \otimes x & \xrightarrow{a_{yzx}} & y \otimes (z \otimes x) \end{array}$$

(3) Inverse law

$$\begin{array}{ccc} & y \otimes x & \\ s_{xy} \swarrow & & \searrow s_{yx} \\ x \otimes y & \xlongequal{\quad} & x \otimes y \end{array}$$

Translation category

Let G be a group or monoid acting on X . Denoted $G \int X$.

Objects: Elements of X .

Morphisms: $\text{Hom}_{G \int X}(x, x') = \{g \in G : g.x = x'\}$

Triangulated category

An **additive category** $\mathcal{K}$ equipped with an automorphism $T : \mathcal{K} \rightarrow \mathcal{K}$ (the translation functor) and distinguished family of triangles (called exact triangles) satisfying axioms (i) through (iv) below. Here a triangle is a triple of morphisms (u, v, w) with $u : A \rightarrow B, v : B \rightarrow C, w : C \rightarrow T(A)$. A morphism of triangles is a triple (f, g, h) s.t. the following commutes:

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\
f \downarrow & & g \downarrow & & h \downarrow & & \downarrow Tf \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A')
\end{array}$$

The category, automorphism, and exact triangles must satisfy the following axioms.

(i) Every morphism $u : A \rightarrow B$ can be embedded in an exact triangle (u, v, w) . If $A = B$ and $C = 0$, then $(\text{id}_A, 0, 0)$ is exact. If (u, v, w) is a triangle on (A, B, C) which is isomorphic to an exact triangle (u', v', w') on (A', B', C') , then (u, v, w) is exact.

(ii) (Rotation) If (u, v, w) is an exact triangle on (A, B, C) , then $(v, w, -Tu), (-T^{-1}u, v, w)$ are exact triangles on $(B, C, TA), (T^{-1}C, A, B)$ (resp.).

(iii) (Morphisms) Given exact triangles (u, v, w) and (u', v', w') with morphisms $f : A \rightarrow A', g : B \rightarrow B'$ such that $gu = u'f$, there exists a morphism $h : C \rightarrow C'$ so that (f, g, h) is a morphism of triangles:

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\
f \downarrow & & g \downarrow & & \exists h \downarrow & & \downarrow Tf \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A')
\end{array}$$

(iv) (Octahedral axiom) Given objects A, B, C, X, Y, Z in K , suppose there are three exact triangles (u, j, ∂) on (A, B, Z) , (v, x, i) on (B, C, X) , and (vu, y, δ) on (A, B, Z) . Then there is a fourth exact triangle $(f, g, T(j)i)$ on (Z, Y, X) such that in the octahedron below, the four exact triangles form four of the faces; $\partial = \delta f, x = gy; yv = fj$; and $u\delta = ig$.

