

Some Notes on Brauer Groups

Samuel Moss

Please note that throughout these notes I will occasionally be lazy with “=” and “ $\cong$.” In particular, I will sometimes use equality when there is an obvious canonical isomorphism. The same applies for “ $\subseteq$ ” and “ $\rightarrow$.”

31/3/2025

Definition 1 (k -algebra)

Let k be a field. Then a k -algebra A is a ring equipped with an embedding $k \hookrightarrow Z(A)$.

Definition 2 (simple rings and algebras)

Let k be a field. A simple k -algebra is a finite-dimensional k -algebra R which is a simple ring.

A ring R is called simple if its only two-sided ideals are R and 0 .

Note that if A, B are k -algebras and A is simple, any k -algebra map is injective. We have that all simple k -algebras are semisimple (this can be viewed as a corollary of the Artin-Wedderburn theorem below).

Artin-Wedderburn Theorems

If R is a semisimple ring, $R \cong \prod_{i=1}^n M_{n_i}(D_i)$ for each D_i a division ring. Then pairs (D_i, n_i) are unique up to isomorphism, and the decomposition is unique up to permutation.

If R is a simple Artinian ring, then $R \cong M_n(D)$ for D a division ring. If R is a simple k -algebra, then $R \cong M_n(D)$ for D a division k -algebra.

Definition 3 (central simple algebra)

A k -algebra A is said to be a central simple algebra (CSA) if $k = Z(A)$ and A is simple.

We will use the notation $[A : k] := \dim_k(A)$.

Definition 4 (quaternion algebra)

Let F be a field. A quaternion F -algebra R is a CSA of dimension 4.

More concretely, let F be a field of characteristic not equal to 2. Then for $a, b \in F^\times$, define $\left(\frac{a, b}{F}\right)$ to be the k -algebra spanned by $\{1, i, j, k\}$ such that $1 \in F \subseteq \left(\frac{a, b}{F}\right)$, $i^2 = a$, $j^2 = b$, and $k = ij = -ji$.

Exercise:

Show that every dimension 4 CSA over a field of characteristic not equal to 2 is of the form described above. Show that the algebra described above is a CSA.

Lemma 1

Let A, B be k -algebras. Then $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$.

Proof:

Note that since we tensor A and B as k -vector spaces, everything is flat, thus we have an inclusion $Z(A) \otimes_k Z(B) \subseteq A \otimes_k B$.

Since multiplication on $A \otimes_k B$ is given by $(a \otimes b)(c \otimes d) = ac \otimes bd$, it is obvious that $Z(A) \otimes_k Z(B) \subseteq Z(A \otimes_k B)$. Thus to show equality, it suffices to show containment in the other direction.

Take $\gamma \in Z(A \otimes_k B)$. Then $\gamma = \sum_{i=1}^n \alpha_i \otimes \beta_i$ for some $\alpha_i \in A, \beta_i \in B$. Furthermore, we may assume the α_i are linearly independent as elements of a k -vector space. From the linearly independence, the k -span of the α_i decomposes as a direct sum. We know that:

$$\left(\bigoplus_{i=1}^n \text{span}_k \{ \alpha_i \} \right) \otimes_k B \cong \bigoplus_{i=1}^n (\text{span}_k \{ \alpha_i \} \otimes_k B)$$

Now suppose $\sum_{i=1}^n \alpha_i \otimes \beta'_i = \sum_{i=1}^n \alpha_i \otimes \beta''_i$. From the isomorphism above and the fact that the direct sum decomposition of a vector is unique, it follows that $\alpha_i \otimes \beta'_i = \alpha_i \otimes \beta''_i$ for each i ; from the definition of a tensor product of vector spaces, it follows that equality holds only when $\beta'_i = \beta''_i$ for each i .

Since $\gamma \in Z(A \otimes_k B)$, for any $b \in B$ we have $(1 \otimes b)\gamma = \gamma(1 \otimes b)$. Expanding, we have:

$$\begin{aligned} (1 \otimes b)\gamma &= (1 \otimes b) \left(\sum_{i=1}^n \alpha_i \otimes \beta_i \right) = \sum_{i=1}^n \alpha_i \otimes b\beta_i \\ \gamma(1 \otimes b) &= \left(\sum_{i=1}^n \alpha_i \otimes \beta_i \right) (1 \otimes b) = \sum_{i=1}^n \alpha_i \otimes \beta_i b \end{aligned}$$

By the comment above, since these are equal we have $b\beta_i = \beta_i b$ for each i . Since $b \in B$ was arbitrary, each $\beta_i \in Z(B)$. From the β_i , we may pick linearly independent β_{i_j} and express $\gamma = \sum_{i=1}^{n'} \alpha'_i \otimes \beta_{i_j}$ for some $\alpha'_i \in A$. Repeating the same argument as above, we find the $\alpha'_i \in Z(A)$, thus $\gamma \in Z(A) \otimes_k Z(B)$, thereby proving $Z(A \otimes_k B) \subseteq Z(A) \otimes_k Z(B)$. ■

In particular, note that if $Z(A) = k$, then $Z(A \otimes_k B) = k \otimes_k Z(B) = Z(B)$. This final “equality” refers to the canonical embedding $Z(B) \hookrightarrow k \otimes_k Z(B)$ given by $b \mapsto 1 \otimes b$.

2/4/2025

Definition 5 (length)

The length of $x \in A \otimes_k B$ is the minimal n s.t. $x = \sum_{i=1}^n a_i \otimes b_i$ for $a_i \in A, b_i \in B$.

Note that the $\{b_i\}$ in the equation above must be linearly independent, for if not we could expand one of the b_ℓ in terms of the other b_k , use the bilinearity of the tensor product, and express x in terms of $n - 1$ simple tensors.

Lemma 2

Let A be a CSA over k , B a simple k -algebra. Then $A \otimes_k B$ is simple.

Proof:

Let $I \subseteq A \otimes_k B$ be a nonzero ideal and take $\gamma \in I \setminus 0$ to be an element of minimal length m in I . Then $\gamma = \sum_{i=1}^m a_i \otimes b_i$ for $\{b_i\}_{i=1}^m$ a set of linearly independent vectors and $a_1 \neq 0$. Because A is simple and a_1 nonzero, $Aa_1A = A$, thus there exist $a'_j, a''_j \in A$ such that $1 = \sum_{j=1}^n a'_j a_1 a''_j$. This in turn implies:

$$\left(\sum_{i=1}^n a'_i \otimes 1 \right) \gamma \left(\sum_{i=1}^n a''_i \otimes 1 \right) = 1 \otimes b_1 + c_2 \otimes b_2 + \dots + c_m \otimes b_m =: \gamma' \in I$$

For some $c_i \in A$. Note that this is nonzero since the b_i are linearly independent and the first simple tensor in the expansion is $1 \otimes b_1$. Now for any $a \in A$, $(a \otimes 1)\gamma' - \gamma'(a \otimes 1) = \sum_{i=2}^m (ac_i - c_i a) \otimes b_i$. If this is nonzero, then we have found a nonzero element of I with length $m - 1$, contradicting the minimality of m . Thus $\sum_{i=2}^m (ac_i - c_i a) \otimes b_i = 0$, and because the b_i are linearly independent, this means $ac_i - c_i a = 0$ for all i , and since $a \in A$ is arbitrary and A is a CSA over k , $c_i \in Z(A) = k$ for each i . Thus $c_i \otimes b_i = 1 \otimes c_i b_i$, so $\gamma' = 1 \otimes b$, where $b = b_1 + \sum_{i=2}^m c_i b_i$.

Because b is simple and nonzero, $BbB = B$, thus there exist $b'_\ell, b''_\ell \in B$ with $\sum_{i=1}^{n'} b'_\ell b b''_\ell = 1$. But this implies:

$$\left(\sum_{i=1}^{n'} 1 \otimes b'_\ell \right) (1 \otimes b) \left(\sum_{i=1}^{n'} 1 \otimes b''_\ell \right) = 1 \otimes \sum_{i=1}^{n'} b'_\ell b b''_\ell = 1 \otimes 1 \in I$$

Thus the unit is in I , so $I = A \otimes_k B$. ■

Theorem 1

Let A, B be k -algebras with A a CSA over k and B a finite-dimensional simple k -algebra. Then $A \otimes_k B$ is a CSA over $Z(B)$.

Proof:

This follows almost immediately from Lemmas 1 and 2; we just need to verify that $A \otimes_k B$ is finite dimensional over $Z(B)$. But since $k \subseteq Z(B) \subseteq A \otimes_k B$ are k -submodules and $[A \otimes_k B : k] < \infty$ (A, B are finite-dimensional k -vector spaces), $[A \otimes_k B : Z(B)] < \infty$. ■

Corollary 1

Let A, B be CSAs over k . Then $A \otimes_k B$ is a CSA over k .

Proof:

This is immediate from Theorem 1 and the fact that since B is a CSA over k , $Z(B) = k$. ■

Lemma 3

Let V be a finite-dimensional vector space over k and K/k any field extension. Then $\dim_K(V \otimes_k K) = \dim_k(V)$.

Proof:

Let $n := \dim_k(V)$ and $\{e_i\}_{i=1}^n$ a k -basis for V . We claim that $\{e_i \otimes 1\}_{i=1}^n$ is a k -basis for $V \otimes_k K$. Indeed, first note that any element $v \in V \otimes_k K$ may be expressed as $v = \sum_{i=1}^\ell v_i \otimes \alpha_i$ for $v_i \in V$ and $\alpha_i \in K$. Expanding each $v_i = \sum_{j=1}^n \beta_{ij} e_j$ for $\beta_{ij} \in k$ and using the k -bilinearity of the tensor product, we find:

$$v = \sum_{i=1}^\ell \left(\sum_{j=1}^n \beta_{ij} e_j \right) \otimes \alpha_i = \sum_{i,j=1}^{\ell,n} e_j \otimes \beta_{ij} \alpha_i = \sum_{j=1}^n e_j \otimes \left(\sum_{i=1}^\ell \beta_{ij} \alpha_i \right) = \sum_{j=1}^n \left(\sum_{i=1}^\ell \beta_{ij} \alpha_i \right) (e_j \otimes 1)$$

Since $\sum_{i=1}^\ell \beta_{ij} \alpha_i \in K$ for each j , we see that $\text{span}_k \{e_i \otimes 1\}_{i=1}^n = V \otimes_k K$.

Now suppose $\sum_{i=1}^n \alpha_i (e_i \otimes 1) = \sum_{i=1}^n e_i \otimes \alpha_i = 0$ for some set of coefficients $\alpha_i \in K$. Then since the e_i are linearly independent over k , we have from the same argument used in Lemma 1 that $\alpha_i = 0$ for each i . Thus the $e_i \otimes 1$ are linearly independent, so $\{e_i \otimes 1\}_{i=1}^n$ is a basis for $V \otimes_k K$, thus $\dim_K(V \otimes_k K) = n = \dim_k(V)$. ■

Corollary 2

Let A be a CSA over k and K/k any field extension. Then $A \otimes_k K$ is a CSA over K .

Proof:

Since K is a field, $Z(K) = K$, so by Lemmas 1 and 2 we have that $A \otimes_k K$ is simple and central over K . Furthermore, by Lemma 3 and the fact that CSAs are finite-dimensional, $[A \otimes_k K : K] = [A : k] < \infty$, thus $A \otimes_k K$ is finite-dimensional and therefore a CSA over K . ■

Corollary 3

Let A be a CSA over k with $\bar{k}$ an algebraic closure of k . Then $A \otimes_k \bar{k} \cong M_n(\bar{k})$ for some $n \in \mathbb{Z}_{\geq 1}$. In particular, $[A : k] = [A \otimes_k \bar{k} : \bar{k}] = n^2$.

Proof:

The last statement follows from Lemma 3 and the first statement.

By Corollary 2, $A \otimes_k \bar{k}$ is a CSA over $\bar{k}$, so by Artin-Wedderburn $A \otimes_k \bar{k} \cong M_n(D)$ for some $n \in \mathbb{Z}_{\geq 1}$ and D a (finite-dimensional) division algebra over $\bar{k}$ (the finite-dimensional condition follows from the fact that $[A : k] = [A \otimes_k \bar{k} : \bar{k}] < \infty$). But we know the only finite-dimensional division algebra over the algebraically closed field $\bar{k}$ is $\bar{k}$ itself, thus $A \otimes_k \bar{k} \cong M_n(\bar{k})$. ■

Definition 6 (centralizer)

Given a ring A and subset $X \subseteq A$, the centralizer $C_A(X) := \{a \in A : ax = xa \ \forall x \in X\}$.

Double Centralizer Theorem

Let A be a CSA over k and $B \subseteq A$ a simple k -algebra. Then:

- (i) $C_A(B)$ is a simple k -algebra.*
- (ii) $B \otimes_k A^{\text{op}} \cong M_s(D)$, $C_A(B) \cong M_r(D^{\text{op}})$ for a division k -algebra D , $r, s \in \mathbb{Z}_{\geq 1}$ and $r \mid s$.*
- (iii) $[B : k][C_A(B) : k] = [A : k]$*
- (iv) $C_A(C_A(B)) = B$*

Proof:

(i) and (ii)

Recall that for a simple Artinian ring R , all minimal left ideals are isomorphic (denote this ideal by L). Then there exists $n \in \mathbb{Z}_{\geq 1}$ with $R \cong L^{\oplus n}$. Additionally, $R^{\text{op}} \cong \text{End}_R(R) \cong M_n(\text{End}_R(L))$, and any simple R -module is isomorphic to L .

Taking R in the discussion above to be $B \otimes_k A^{\text{op}}$ (which is simple by Theorem 1), we find that there exists $s \in \mathbb{Z}_{\geq 1}$ such that $B \otimes_k A^{\text{op}} \cong M_s(D)$, where $D^{\text{op}} \cong \text{End}_{B \otimes_k A^{\text{op}}}(L)$ and $B \otimes_k$

$A^{\text{op}} \cong L^{\oplus s}$. Note that A is a $B \otimes_k A^{\text{op}}$ -module, the action given by $(b \otimes a)x = bxa$, thus $A \cong L^{\oplus r}$ for some $r \in \mathbb{Z}_{\geq 1}$.

We claim that $C_A(B) \cong \text{End}_{B \otimes_k A^{\text{op}}}(A) \cong M_r(D)$. The latter isomorphism is immediate. Define the map $\text{End}_{B \otimes_k A^{\text{op}}}(A) \rightarrow A$ by $\varphi \mapsto \varphi(1)$. Suppose $\varphi(1) = 0$. Then by definition of the $B \otimes_k A^{\text{op}}$ action, $\varphi(a) = \varphi(1)a$ for all $a \in A$, thus $\varphi(1) = 0 \implies \varphi(a) = \varphi(1)a = 0$ for all $a \in A$, so $\varphi = 0$, thus this map is injective. Because φ is a $(B \otimes_k A^{\text{op}})$ -module map, for any $b \in B, a, x \in A$ we have $b\varphi(x)a = \varphi(bxa)$, thus $b\varphi(1)xa = \varphi(1)bxa$. In particular, this is true when $a = x = 1$, thus $b\varphi(1) = \varphi(1)b$ for each $b \in B$, so $\varphi(1) \in C_A(B)$. On the other hand, given $y \in C_A(B)$, the map determined by $\varphi(1) = y$ is easily checked to be a $(B \otimes_k A^{\text{op}})$ -module map. Thus the map $\text{End}_{B \otimes_k A^{\text{op}}}(A) \rightarrow C_A(B)$ is an isomorphism.

Now note that:

$$[A : k] = s[L : k], [B \otimes_k A^{\text{op}} : k] = [B : k][A : k] = r[L : k] \implies \frac{s[L : k]}{r[L : k]} = \frac{s}{r} = [B : k]$$

Thus $r \mid s$.

(iii)

From (i) and (ii):

$$[C_A(B) : k] = [M_r(D) : k] = r^2[D : k], [B : k][A : k] = [B \otimes_k A^{\text{op}} : k] = s^2[D : k]$$

Dividing these and using that $\frac{s}{r} = [B : k]$:

$$\frac{[C_A(B) : k]}{[B : k][A : k]} = \frac{r^2[D : k]}{s^2[D : k]} = \frac{r^2}{s^2} = \frac{1}{[B : k]^2} \implies [C_A(B) : k][B : k] = [A : k]$$

(iv)

In any case we have $B \subseteq C_A(C_A(B))$. To see why, take $b \in B$ and $a \in C_A(B)$. Then $ab = ba$ by definition of $C_A(B)$, thus $ab = ba$ for any $a \in C_A(B)$, thus $b \in C_A(C_A(B))$. It suffices then to show that the k -dimensions of $C_A(C_A(B))$ and B are the same. Applying (iii) twice, the first time replacing B with $C_A(B)$:

$$[C_A(C_A(B)) : k] = \frac{[A : k]}{[C_A(B) : k]} = \frac{[A : k]}{\frac{[A : k]}{[B : k]}} = [B : k]$$

■

4/4/2025

Definition 7 (similar)

Let A, B be CSAs over a field k . Then A is similar to B , denoted $A \sim B$, if there exist $m, n \in \mathbb{Z}_{\geq 1}$ such that $A \otimes_k M_m(k) \cong B \otimes_k M_n(k)$.

Lemma 4

For a field k and $m, n \in \mathbb{Z}_{\geq 1}$, $M_n(k) \otimes_k M_m(k) \cong M_{mn}(k)$. More generally, let V, W be finite-dimensional k -vector spaces. Then $\text{End}_k(V) \otimes_k \text{End}_k(W) \cong \text{End}_k(V \otimes_k W)$.

Proof:

Since V, W are finite-dimensional, $V \cong k^{\oplus m}$ and $W \cong k^{\oplus n}$ for some $m, n \in \mathbb{Z}_{\geq 1}$. We know from linear algebra that $V \otimes W \cong k^{\oplus mn}$, thus $\text{End}_k(V) \cong M_m(k)$, $\text{End}_k(W) \cong M_n(k)$, and $\text{End}_k(V \otimes W) \cong M_{mn}(k)$. Counting dimension:

$$\begin{aligned} \dim_k(\text{End}_k(V) \otimes_k \text{End}_k(W)) &= \dim_k(\text{End}_k(V)) \dim_k(\text{End}_k(W)) \\ &= \dim_k(M_m(k)) \dim_k(M_n(k)) = m^2 n^2 \end{aligned}$$

$$\dim_k(\text{End}_k(V \otimes_k W)) = \dim_k(M_{mn}(k)) = m^2 n^2$$

Thus $\text{End}_k(V) \otimes_k \text{End}_k(W)$ and $\text{End}_k(V \otimes_k W)$ have the same k -dimension.

There is an obvious map given by:

$$\Phi : \text{End}_k(V) \otimes_k \text{End}_k(W) \rightarrow \text{End}_k(V \otimes_k W)$$

$$\Phi(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$$

Note that:

$$\begin{aligned} \Phi((f \otimes g) \circ (f' \otimes g'))(v \otimes w) &= \Phi((f \circ f') \otimes (g \circ g'))(v \otimes w) = (f \circ f')(v) \otimes (g \circ g')(w) \\ &= f(f'(v)) \otimes g(g'(w)) = \Phi(f \otimes g)(f'(v) \otimes g'(w)) \\ &= (\Phi(f \otimes g) \circ \Phi(f' \otimes g'))(v \otimes w) \end{aligned}$$

And:

$$\begin{aligned} \Phi(k(f \otimes g))(v \otimes w) &= \Phi((kf) \otimes g)(v \otimes w) = (kf)(v) \otimes g(w) \\ &= k(f(v) \otimes g(w)) = k\Phi(f \otimes g)(v \otimes w) \end{aligned}$$

thus Φ is a map of k -algebras. Since the domain and codomain have the same dimension and k -algebra maps are also maps of k -vector spaces, to prove Φ is an isomorphism it suffices to show that Φ is surjective. Consider a map $\varphi : \text{End}_k(V \otimes_k W) \rightarrow \text{End}_k(V \otimes_k W)$. This map is determined by $\varphi(e_i \otimes f_j)$ for $\{e_i\}$ a basis of V and $\{f_j\}$ a basis of W . Furthermore, there exist $\alpha_{ijk\ell} \in k$ such that $\varphi(e_i \otimes f_j) = \sum_{k,\ell} \alpha_{ijk\ell} e_k \otimes e_\ell$. Define the map $E_{i\mu} : V \rightarrow V$ by $E_{i\mu}(e_j) = \delta_{ij} e_\mu$ and $F_{i\mu} : W \rightarrow W$ by $F_{i\mu}(f_j) = \delta_{ij} f_\mu$. Then note that:

$$\begin{aligned} \Phi\left(\sum_{\mu,\nu,k,\ell} \alpha_{\mu\nu k\ell} E_{\mu k} \otimes F_{\nu \ell}\right)(e_i \otimes f_j) &= \sum_{\mu,\nu,k,\ell} \alpha_{\mu\nu k\ell} E_{\mu k}(e_i) \otimes F_{\nu \ell}(f_j) \\ &= \sum_{\mu,\nu} \sum_{k,\ell} \alpha_{\mu\nu k\ell} \delta_{\mu i} e_k \otimes \delta_{\nu j} e_\ell \\ &= \sum_{k,\ell} \alpha_{ijk\ell} e_k \otimes e_\ell \end{aligned}$$

Thus this function is equal to φ . Since $\sum_{\mu,\nu,k,\ell} \alpha_{\mu\nu k\ell} E_{\mu k} \otimes F_{\nu \ell} \in \text{End}_k(V) \otimes_k \text{End}_k(W)$, it follows that Φ is surjective, as desired. ■

We claim that similarity is an equivalence relation. Reflexivity and symmetry are clear. For transitivity, suppose that A, B, C are CSAs over k with $A \sim B, B \sim C$. Then there exist $m, n, r, s \in \mathbb{Z}_{\geq 1}$ such that:

$$A \otimes_k M_m(k) \cong B \otimes_k M_n(k), B \otimes_k M_r(k) \cong C \otimes_k M_s(k)$$

So by Lemma 4 and the commutativity of tensor products:

$$\begin{aligned}
A \otimes_k M_{mr}(k) &\cong A \otimes_k M_m(k) \otimes_k M_r(k) \cong B \otimes_k M_n(k) \otimes_k M_r(k) \\
&\cong C \otimes_k M_n(k) \otimes_k M_s(k) \cong C \otimes_k M_{ns}(k)
\end{aligned}$$

From which it is clear $A \sim C$.

Definition 8 (Brauer group)

The Brauer group of a field k is defined as the set $\text{Br}(k) := \{[A] : A \text{ is a CSA over } k\}$ with the group multiplication given by $[A][B] = [A \otimes_k B]$. Here $[A]$ is the equivalence class under similarity.

By commutativity of the tensor product, the Brauer group is abelian. We need to check that (1) the product is well-defined, (2) there exists an identity, (3) there exist inverses.

Well-definedness:

Suppose $A \sim A'$ and $B \sim B'$. Then there exist $m, n, r, s \in \mathbb{Z}_{\geq 1}$ such that:

$$A \otimes_k M_m(k) \cong A' \otimes_k M_n(k), B \otimes_k M_r(k) \cong B' \otimes_k M_s(k)$$

Indeed, by Lemma 4 and commutativity of $\otimes_k$:

$$\begin{aligned}
A \otimes_k B \otimes_k M_{mr}(k) &\cong A \otimes_k M_m(k) \otimes_k B \otimes_k M_r(k) \\
&\cong A' \otimes_k M_n(k) \otimes_k B' \otimes_k M_s(k) \\
&\cong A' \otimes_k B' \otimes_k M_{ns}(k)
\end{aligned}$$

Thus $A \otimes_k B \sim A' \otimes_k B'$.

Identity:

Because k is a CSA over itself and $A \otimes_k k \cong A$ for any k -algebra A , $[A][k] = [A \otimes_k k]$ for all $[A] \in \text{Br}(k)$, thus $[k]$ is the identity element.

Inverses:

Take $[A] \in \text{Br}(k)$ and consider $A \otimes_k A^{\text{op}}$. By Artin-Wedderburn, $A \otimes_k A^{\text{op}} \cong M_n(D^{\text{op}})$ for some $n \in \mathbb{Z}_{\geq 1}$ and division k -algebra D . Since A is a CSA, $C_A(A) = Z(A) = k$, so by the double centralizer theorem $k = C_A(A) \cong M_r(D)$ and $A \otimes_k A^{\text{op}} \cong M_n(D^{\text{op}})$, thus $D = D^{\text{op}} = k$, so $A \otimes_k A^{\text{op}} \cong M_n(k)$. Since $[M_n(k)] = [k]$, $[A][A^{\text{op}}] = [A \otimes_k A^{\text{op}}] = [M_n(k)] = [k]$. Since $\text{Br}(k)$ is commutative, it follows that $[A^{\text{op}}]$ is the two-sided inverse of $[A]$.

Proposition 1

Given a field k , $\text{Br}(k) \leftrightarrow \{\text{division } k\text{-algebras with center } k\}$, the correspondence being given by $[D] \leftrightarrow D$ in one direction and Artin-Wedderburn in the other.

Proof:

First note that if A is a CSA over k , then by Artin-Wedderburn $A \cong M_n(D)$ for some $n \in \mathbb{Z}_{\geq 1}$ and D a division k -algebra. Since A is central, we also have $Z(D) = k$.

Consider the map $\Psi : D \otimes_k M_n(k) \rightarrow M_n(D)$ given by $\Psi(x \otimes m)_{ij} = xm_{ij}$. It is clear this map is k -linear. To show it is a ring map:

$$\Psi(xy \otimes mm')_{ij} = xy(mm')_{ij} = xy \sum_{k=1}^n m_{ik}m'_{kj}$$

$$\begin{aligned}
(\Psi(x \otimes m)\Psi(y \otimes m'))_{ij} &= \sum_{k=1}^n \Psi(x \otimes m)_{ik} \Psi(y \otimes m')_{kj} \\
&= \sum_{k=1}^n x m_{ik} y m'_{kj} = xy \sum_{i=1}^n m_{ik} m'_{kj}
\end{aligned}$$

Counting dimension, $[D \otimes_k M_n(k) : k] = [D : k][M_n(k) : k] = n^2[D : k] = [M_n(D) : k]$, thus to show Ψ is an isomorphism it suffices to prove either injectivity or surjectivity. But surjectivity is easy; given a matrix $m \in M_n(D)$, $m = \sum_{i,j=1}^{n,n} \Psi(m_{ij} \otimes e_{(ij)})$, where $e_{(ij)} \in M_n(k)$ is the matrix with the (i, j) -entry one and all others zero. Thus $D \otimes_k M_n(k) \cong M_n(D)$, so each element of $\text{Br}(k)$ may be represented by some central division k -algebra D . To prove the claim, it suffices to show that each element of $\text{Br}(k)$ is represented by a *unique* division algebra, i.e. $[D] = [D'] \implies D \cong D'$. Suppose $[D] = [D']$. Then there exists some $m, n \in \mathbb{Z}_{\geq 1}$ such that $D \otimes_k M_m(k) \cong D' \otimes_k M_n(k)$. As shown above, this means $M_m(D) \cong M_n(D')$. But by the uniqueness of Artin-Wedderburn this implies $D = D'$. ■

Noether-Skolem Theorem

Let A be a CSA over k and B a simple k -algebra. Consider maps $f, g : B \rightarrow A$ (note that these maps must be injective since B is simple). Then there exists some $u \in A^\times$ such that for all $b \in B$, $g(b) = uf(b)u^{-1}$.

Proof:

From Theorem 1, $B \otimes A^{\text{op}}$ is a finite-dimensional simple k -algebra, thus all simple finite-dimensional modules are isomorphic. In particular, any two finite-dimensional modules are isomorphic.

Let ${}_f A$ and ${}_g A$ be left $(B \otimes A^{\text{op}})$ -modules with the left B -action given by f, g (resp.) and the right A -action that of A . Since these have the same underlying k -action, $[{}_f A : k] = [{}_g A : k]$, thus ${}_f A \cong {}_g A$. Let $\varphi : {}_f A \rightarrow {}_g A$ be such an isomorphism. Then for any $b \otimes a \in B \otimes A^{\text{op}}$ and $x \in A$:

$$\varphi((b \otimes a)_f x) = (b \otimes a)_g \varphi(x) \implies \varphi(f(b)xa) = g(b)\varphi(x)a$$

In particular, it holds when $b = 1, x = 1$, thus $\varphi(a) = \varphi(1)a$ for all $a \in A$. This implies:

$$\varphi(f(b)xa) = \varphi(1)f(b)xa = g(b)\varphi(x)a = g(b)\varphi(1)xa$$

Again, this holds when $x = a = 1$. Define $u := \varphi(1)$. Then for all $b \in B$, $uf(b) = g(b)u$.

Because φ is an isomorphism, φ^{-1} exists. Then $\varphi^{-1}\varphi(1) = \varphi^{-1}(u) = 1$. But by the same argument as above, $\varphi^{-1}(u) = \varphi^{-1}(1)u$, so $\varphi^{-1}(1)u = 1$. It is easy to verify that $u\varphi^{-1}(1) = 1$ as well, thus $u \in A^\times$. ■

In the following corollary, why do we need the map to be an isomorphism?

Corollary 4

Let A be a CSA over k and $B_1, B_2 \subseteq A$ simple k -subalgebras. If $\varphi : B_1 \rightarrow B_2$ is an isomorphism, then there exists $u \in A^\times$ such that for all $b \in B_1$, $\varphi(b) = ubu^{-1}$.

Proof:

Let $i_1 : B_1 \hookrightarrow A, i_2 : B_2 \hookrightarrow A$ be the canonical inclusions. Then by Noether-Skolem, there exists some $u \in A^\times$ such that $i_1(b) = u(i_2 \circ \varphi)(b)u^{-1}$ for all $b \in B_1$. Rewriting, we see that for all $b \in B_1$, $\varphi(b) = i_2\varphi(b) = u^{-1}i_1(b)u = u^{-1}bu$, proving the claim. ■

Corollary 5

Let A be a CSA over k . Then $\text{Aut}_{k\text{-alg}}(A) \cong A^\times/k^\times$.

Proof:

In Corollary 4, let $B_1 = B_2 = A$; then any $\varphi \in \text{Aut}_{k\text{-alg}}(A)$ is conjugate to the identity automorphism, i.e., $\varphi(a) = uau^{-1}$ for some $u \in A^\times$. On the flip side, for any map $u \in A^\times$, the map $a \mapsto uau^{-1}$ is an automorphism. Thus we have a surjective map $A^\times \twoheadrightarrow \text{Aut}_{k\text{-alg}}(A)$ which can be easily checked to be one of rings. Let $\varphi_u(a) = uau^{-1}$. Then $\varphi_u = \text{id}_A$ if $uau^{-1} = a$ for all $a \in A$, i.e. $ua = au$, i.e. $u \in Z(A) \cap A^\times$. Since A is central over k , this means $u \in k^\times$, thus the kernel of the aforementioned map is $k^\times$, so by the first isomorphism theorem $\text{Aut}_{k\text{-alg}}(A) \cong A^\times/k^\times$. ■

Definition 9 (splitting)

A splitting field for A a CSA over k is a field extension K/k such that $A \otimes_k K \cong M_n(K)$. This is equivalent to the statement that $[A \otimes_k K] = [K]$ in $\text{Br}(K)$.

Note that we can characterize Brauer equivalence as follows: $A \sim B$ iff $A \cong M_m(D), B \cong M_n(D)$ for $m, n \in \mathbb{Z}_{\geq 1}$ and D a division k -algebra. On the one hand, it is clear that $M_m(D) \sim M_n(D)$ since $M_m(D) \otimes_k M_n(k) \cong M_n(D) \otimes_k M_m(k)$. On the other hand, if $A \sim B$, then $A \otimes_k M_m(k) \cong B \otimes_k M_n(k)$ for some $m, n \in \mathbb{Z}_{\geq 1}$. By Artin-Wedderburn, $A \cong M_r(D), B \cong M_s(D')$ for some $r, s \in \mathbb{Z}_{\geq 1}$ and D, D' division k -algebras. Thus $M_{rm}(D) \cong M_{ns}(D')$. By the uniqueness in Artin-Wedderburn, $D \cong D'$ (and $rm = ns$).

Now the equivalence of definitions is obvious: $[A \otimes_k K] = [K]$ iff $A \otimes_k K \cong M_n(K)$.

Examples:

Let k be a field with algebraic closure $\bar{k}$. Then since $\bar{k}$ has no finite-dimensional division algebras over it, when A is a CSA over k we have by Artin-Wedderburn $A \otimes_k \bar{k} \cong M_n(\bar{k})$ for some $n \in \mathbb{Z}_{\geq 1}$, thus $\bar{k}$ is a splitting field for A .

Let k be a field of characteristic not equal to two and consider the quaternion algebra $A = \left(\frac{a,b}{k}\right), a, b \in k^\times$. If $a \in (k^\times)^2$, then $A \cong M_2(k)$, the isomorphism being given by:

$$i \mapsto \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

Thus k is a splitting field for A . If $a \notin (k^\times)^2$, then $k(\sqrt{a})$ is a splitting field for A , the isomorphism $A \otimes_k k(\sqrt{a}) \rightarrow M_2(k(\sqrt{a}))$ given by:

$$i \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

Add in commentary about the subtleties of point (3).

Lemma 5

Let A be a CSA over k and let L be a subfield $k \subseteq L \subseteq A$. Then the following are equivalent:

- (1) $C_A(L) = L$, (2) $[A : k] = [L : k]^2$, (3) L is the maximum commutative k -subalgebra of A

Proof:

$$(1) \implies (2)$$

Because L is a field, it is a simple k -algebra. Thus by DCT, $[L : k][C_A(L) : k] = [A : k]$. By the (1), the dimensions are equal, thus $[L : k]^2 = [A : k]$.

$$(2) \implies (3)$$

Suppose L is not maximal. Then there exists some $L' \supsetneq L$ which is commutative. Taking these two facts together, it is clear $L' \subseteq C_A(L)$ and that $[L' : k] > [L : k]$. Thus $[C_A(L) : k] \geq [L' : k] > [L : k]$, so (2) cannot hold.

$$(3) \implies (1)$$

If $L \subsetneq C_A(L)$, then there exists some $\gamma \in C_A(L) \setminus L$, thus $L[\gamma] \supsetneq L$ is a commutative k -subalgebra. But this contradicts (3) since L was assumed to be the largest possible commutative k -subalgebra, so it must be $L = C_A(L)$. ■

Corollary 6

Let D be a finite-dimensional division k -algebra with $Z(D) = k$. Then every maximum subfield of D has degree $\sqrt{[D : k]}$ over k .

Proof:

Every commutative k -subalgebra of D is a field. The result follows from the equivalence of (3) and (2) of Lemma 5. ■